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Deformation and breakup of a single slender drop in an extensional flow

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The deformation and conditions for breakup of a single slender drop placed symmetrically in a uniaxial extensional flow are examined theoretically. For the case of an inviscid drop in zero-Reynolds-number flow, Buckmaster (1972) showed, using slender-body analysis, that the shape of the drop is given by $r \equiv \epsilon R(z) = \epsilon(1-|z|^{\nu})/2\nu$, where $\epsilon \equiv \gamma/G\mu l$ and γ is the interfacial tension, G the strength of the extensional flow, μ the viscosity of the suspending fluid and l the drop half-length; also $\nu = \frac{1}{2}P-1$, where P is the unknown constant pressure inside the drop rendered dimensionless with respect to $G\mu$. By requiring that R be analytic at $z = 0$, Buckmaster then concluded that ν had to be an even integer and thereby obtained a countably infinite set of slender profiles for any (large) value of the flow strength G. In the present work, the expression for $R(z)$ shown above is obtained readily using the method of inner and outer expansions, the method failing when $|z| \leqslant O(\epsilon)$ and ν is not an even integer. Thus, in general, a new solution is needed to describe the shape within the 'singular' region $|z| \leqslant O(\epsilon)$. The requirement that the two solutions match in their domain of overlap then leads to the conclusion that ν can be either equal to 2 or greater than or equal to 3. However, a stability analysis reveals that only the solution with $\nu = 2$ is stable, and hence a unique shape exists.

Next, drops of low viscosity $\mu_i = O(\epsilon^2\mu)$ are examined in zero-Reynolds-number flow. Here, again, a unique solution is obtained according to which a steady shape cannot exist if $(G\mu a/\gamma)(\mu_i/\mu)^{\frac{1}{6}} > 0{\cdot}148$, where $a \equiv (3V/4\pi)^{\frac{1}{3}}$ and V is the volume of the drop. This breakup criterion is identical to that found by Taylor (1964). A similar analysis for the case of an inviscid drop in a flow with non-zero Reynolds number shows that drop breakup will occur if $(G\mu a/\gamma)(\rho a\gamma/\mu^2)^{\frac{1}{5}} > 0{\cdot}284$, where ρ is the density of the suspending fluid. Finally, when $\mu_i = O(\epsilon^2\mu)$ and inertial effects are neglected within the drop but retained in the surrounding fluid, the critical value of $(G\mu a/\gamma)(\rho a\gamma/\mu^2)^{\frac{1}{5}}$ required for drop breakup is found as a function of the dimensionless group

$$(\rho a\gamma/\mu^2)^{\frac{1}{5}}(\mu/\mu_i)^{\frac{1}{6}},$$

which depends only on the physical properties of the system and the size of the drop. These last two results are the first which take into account inertial effects in determining the deformation and breakup conditions of a drop placed in a shear field.

1. Introduction

It has been known for some time that, when freely suspended in a fluid of equal density undergoing steady shear, a single drop will deform into a steady non-spherical shape, but that, under some conditions, drop breakup will occur if G, the strength of

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the applied rate of strain, is increased beyond a critical value G_c. The extent of this deformation and the point of breakup will depend on the type of shear flow generated in the experimental apparatus plus the three independent dimensionless groups: $G\mu a/\gamma$, $\lambda \equiv \mu_i/\mu$ and the particle Reynolds number $G\rho a^2/\mu$, where γ is the interfacial tension, a is the radius of the equivalent sphere, i.e. $a \equiv (3V/4\pi)^{\frac{1}{3}}$ with V the volume of the particle, and μ and μ_i are, respectively, the viscosity of the suspending fluid and of the drop.

Quantitative experiments in a simple shear and in a hyperbolic flow have been reported by Taylor (1934), by Rumscheidt & Mason (1961), by Grace (1971) and by Torza, Cox & Mason (1972) under conditions of vanishingly small particle Reynolds numbers, so that the number of independent dimensionless groups reduces to two, i.e. $G\mu a/\gamma$ and λ. Even in these simple cases, though, the behaviour of the drops was found to depend in a complicated manner on the nature of the experiment and on λ. For example, it was determined that, in a simple shear, drops with a viscosity ratio λ greater than about 4 retained a steady shape even when G was increased seemingly without bound, whereas in a hyperbolic flow the same drops broke up when $G\mu a/\gamma$ exceeded a value of approximately 0·4. On the other hand, when $\lambda \ll 1$, the drops became long and slender prior to breakup for both types of shear. This required relatively large strain rates, the critical value of $G\mu a/\gamma$ being proportional to $\lambda^{-0.55}$ in a simple shear† and proportional to $\lambda^{-0.16}$ in a hyperbolic flow (Grace 1971). No corresponding experiments appear to have been performed in which the applied shear is of a more general type or in which the particle Reynolds number lies beyond the creeping-flow regime.

This deformation of individual drops, as well as the maximum rate of strain that they are able to sustain before they will break (or the maximum size that a drop can attain at a given rate of strain), has important implications in a variety of seemingly diverse topics, for example in the design of mixing devices for dispersing one liquid phase into another (Grace 1971) or in the rheology of emulsions (Barthès-Biesel & Acrivos 1973*a*). Thus there exists a need for a theory which can account quantitatively for the experimental observations referred to above and then be used to predict the behaviour of such drops under more general conditions.

To date, theoretical studies on the subject have also been restricted to cases of vanishingly small particle Reynolds numbers. The analysis is most straightforward when the shear is weak, for then the drop is almost spherical and, when $\lambda \leqslant O(1)$, the solution to the creeping-flow equations with the appropriate boundary conditions can be obtained via a regular perturbation expansion in the small parameter $G\mu a/\gamma$. (An analogous expansion in the small parameter λ^{-1} applies when $\lambda \gg 1$ and $G\mu a/\gamma$ is $O(1)$, but this case will not concern us here.) To $O(G\mu a/\gamma)$ the drop is found to deform into an ellipsoid (Taylor 1932; Cox 1969) and hence such a first-order steady-state theory cannot account for the phenomenon of bursting. When the $O(G\mu a/\gamma)^2$ terms are included in the analysis, however, and the series is truncated at this point (Barthès-Biesel & Acrivos 1973*b*), the resulting expression for the deformation suggests that no steady-state shape can be attained when $G\mu a/\gamma$ lies beyond a critical value which depends on λ and on the form of the applied shear. According to this second-order analysis, therefore, the phenomenon of breakup is identified with the non-existence

† The exponent was incorrectly stated as $-1{\cdot}3$ in Barthès-Biesel & Acrivos (1973*b*, p. 17).

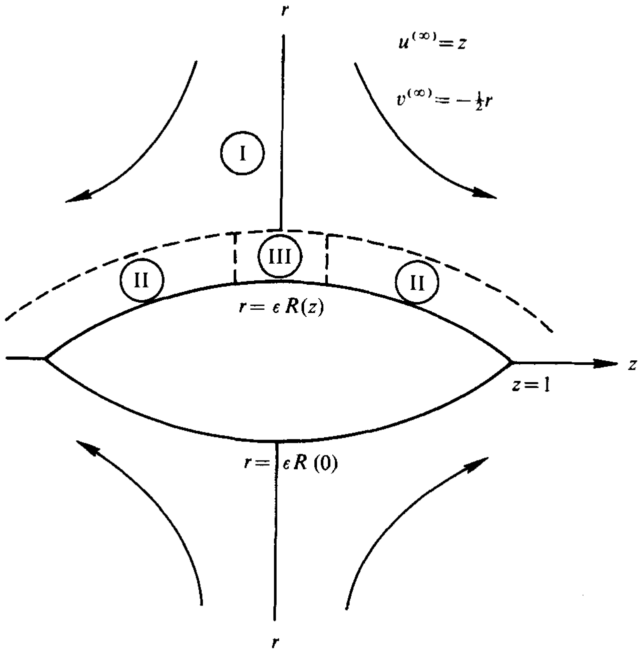

FIGURE 1. The three regions outside a slender drop in an axisymmetric shear field: I, outer region, $r = O(1), z = O(1)$; II, inner region, $r = O(\epsilon), z = O(1)$; III, singular region, $r = O(\epsilon), z = O(\epsilon)$.

of a steady-state solution when $G\mu a/\gamma$ exceeds a critical value, rather than, as is commonly the case in many branches of fluid mechanics, with the instability of the solution to the corresponding steady-state problem when the value of the appropriate dimensionless group, here $G\mu a/\gamma$, lies beyond the critical point.

Using this criterion and their solution truncated to $O(G\mu a/\gamma)^2$, Barthès-Biesel & Acrivos (1973*b*) computed the limiting deformation as well as the critical value of $G\mu a/\gamma$ at breakup, and found that their predictions were generally in good agreement with the experimental data referred to earlier provided that λ was not too small. On the other hand, as was mentioned earlier, the drops are observed to be long and slender prior to breakup when $\lambda \ll 1$ (Grace 1971; Torza *et al.* 1972), and hence the theory by Barthès-Biesel & Acrivos (1973*b*), which expands the solution about the spherical shape and retains only two terms in the series, becomes correspondingly less accurate. In fact, in at least one case, that of an inviscid drop ($\lambda = 0$) freely suspended in an extensional uniaxial straining flow, this theory fails in that it yields a finite critical value for $G\mu a/\gamma$ whereas, as we shall see below, inviscid drops can exist for all $G\mu a/\gamma$.

For drops of low viscosity, i.e. for $\lambda \ll 1$, it would seem much more promising, therefore, to develop a theory that would take advantage of the observed slenderness of the drop prior to breakup, rather than to attempt an extension of the small deformation analysis to higher order in $G\mu a/\gamma$. This was perceived by Taylor (1964), who was apparently the first to propose using the technique of slender-body theory for this purpose and who, by means of an approximate but surprisingly accurate analysis, obtained a quantitative criterion for the maximum value of $G\mu a/\gamma$ for which a steady slender drop can exist under creeping-flow conditions. Recently, Buckmaster (1972, 1973) presented a mathematically systematic and detailed treatment of this general problem which led to a number of very interesting and significant results.

In the first of these papers, Buckmaster (1972) studied the deformation of an inviscid drop in an axisymmetric straining (extensional) motion and showed, using slender-body theory, that for large values of the shear rate the shape of the drop is given asymptotically by

$$r \equiv \epsilon R(z) = (\epsilon/2\nu)(1-|z|^{\nu}), \tag{1.1}$$

where r and z are, respectively, the radial and axial co-ordinates (see figure 1) rendered dimensionless with respect to l, the half-length of the drop, ϵ (assumed small) is defined as $\gamma/G\mu l$ and $\nu = \frac{1}{2}P-1$, P being the difference between the constant pressure inside the inviscid drop and that of the undisturbed flow, both rendered dimensionless with respect to $G\mu$. Of course, since P and therefore ν are *a priori* unknown, the solution as given by (1.1) is incomplete and no further information on ν can be obtained even when the analysis is extended to higher orders in ϵ (Buckmaster 1972). By imposing the additional requirement that $R(z)$ be analytic near $z = 0$, however, Buckmaster concluded that ν had to be an even positive integer with $\nu = 2$ being the most probable choice since, among all such solutions, it corresponds to the drop with the smallest deformation. Indeed, by solving the creeping-flow equations for this case numerically, Youngren & Acrivos (1976) were able to confirm that, when an inviscid drop is progressively elongated from its initial spherical shape after a steady increase in the strength of the applied shear, its shape conforms asymptotically to (1.1) with $\nu = 2$. This numerical study was unable, however, to yield any information concerning the branches of the solution associated with $\nu = 4, 6, \ldots$, except, perhaps, that they could not be attained by any process in which the drop is continuously deformed starting from rest.

At any rate, it is evident from (1.1) and from the numerical work of Youngren & Acrivos (1976) that, in the absence of inertial effects, an inviscid drop can in principle attain a steady shape when placed in an extensional flow, no matter how large the strength of the applied shear. However, when λ is not identically zero but is chosen, for reasons given in § 3, as $O(\epsilon^2)$, Buckmaster (1973) showed in agreement with Taylor's 1964) earlier result that the drop can still deform into a steady slender shape, but that steady solutions to the creeping-flow equations satisfying the relevant boundary conditions do not exist if G exceeds a critical value G_c. We recall that this non-existence of steady shapes for $G > G_c$ is consistent with the findings of Barthès-Biesel & Acrivos (1973*b*) for the case $\lambda \leqslant O(1)$ and thus appears to be a characteristic property of all single drops freely suspended in a steady shearing motion provided that the viscosity ratio λ is non-zero.

In the present paper we shall examine in more detail the deformation of a single drop freely suspended in a steady axisymmetric straining (extensional) motion under conditions when the drop is slender. First of all, for the case $\lambda = 0$, we shall derive (1.1) using the method of inner and outer expansions, which, as will be seen, is much easier to apply than the method employed by Buckmaster (1972). We shall show next that, in general, the inner solution, which leads directly to (1.1), no longer applies within the singular region $z \leqslant O(\epsilon)$, where a different solution must be developed. The requirement that these two solutions match within the domain of overlap then provides a condition for determining ν, which is found to be an even integer (although the proof is incomplete for $\nu \geqslant 3$). Thus, although we reach the same conclusion as Buckmaster (1972) concerning the permissible values of ν, the arguments leading to the final result appear to be more convincing.

We shall next consider the stability of these solutions and shall show, on the basis again of the creeping-flow equations but with the time-dependent term retained in the kinematic condition, that they are all unstable except that for $\nu = 2$. Thus a unique solution to the original problem is shown to exist. Next we shall turn briefly to the case $\lambda = O(\epsilon^2)$ already treated by Buckmaster (1973). Our solution is the same as his, but by re-examining its implications, we shall arrive at a somewhat different, and more correct, expression for the critical shear rate G_c which, in fact, is identical to that given by Taylor (1964).

Following this we shall extend our development to include the effects of inertia. This does not appear to have been done before. It is, however, an important point to consider, because drops of low viscosity become long and slender prior to breakup and hence the Reynolds number based on length could often be significant. First, we shall examine the case of negligible internal viscosity and density, i.e. that of a gas bubble, and obtain a solution, valid for all fluid Reynolds numbers, which is (asymptotically) exact provided only that the bubble is sufficiently slender. Again, as in the case of vanishing fluid inertia, the resulting expression for the shape of the bubble contains an unknown parameter ν which can take on a countably infinite number of values. The latter are no longer even integers but depend on the Reynolds number. In view of our stability results for zero Reynolds number, however, we shall take it for granted that only the lowest branch of the solution, i.e. that corresponding to the smallest drop deformation, is stable. Here, though, in contrast to (1.1), steady bubble shapes are possible only if the applied shear rate does not exceed a critical value G_c for which an expression is given. Finally, an analysis will be presented for the case of an inertialess drop with viscosity ratio $\lambda = O(\epsilon^2)$ freely suspended in a fluid with non-zero Reynolds number, and again a critical shear rate G_c will be shown to exist.

It is felt, therefore, that the results to be presented below significantly extend our understanding of drop deformation and breakup and, for the first time, provide a quantitative measure of the effects of fluid inertia on this phenomenon.

2. An inviscid drop ($\lambda = 0$) in zero-Reynolds-number flow

2.1. *The flow in the inner region*

We consider an inviscid drop symmetrically placed in a uniaxial straining flow. If the velocity components in cylindrical co-ordinates (z, r, ϕ) are denoted by (u, v, w), the non-dimensional undisturbed velocity is

$$u^{(\infty)} = z, \quad v^{(\infty)} = -\tfrac{1}{2}r, \quad w^{(\infty)} = 0. \tag{2.1}$$

Also, the undisturbed pressure $p^{(\infty)}$ will be set equal to zero. Initially, all distances will be rendered dimensionless by l, the unknown half-length of the drop, all velocities by Gl, where G denotes the strength of the applied strain rate, and all stresses by $G\mu$, where μ is the viscosity of the suspending fluid. Also, we let the equation for the shape of the drop be

$$r = \epsilon R(z), \tag{2.2}$$

where $R(z)$ is an $O(1)$ quantity and ϵ is a parameter proportional to the slenderness ratio of the drop, which we shall assume to be small. We shall further suppose that the Reynolds number of the motion is negligibly small, and hence that the creeping-flow equations remain valid throughout the flow field.

We shall develop our solution for small ϵ by the method of inner and outer expansions, which has already been applied to zero-Reynolds-number flows past *solid* slender particles by Batchelor (1970) and, more recently, by Keller & Rubinow (1976). The analysis in the present case is particularly simple because, to a first approximation in ϵ, an inviscid drop will alter the undisturbed velocity field only within the inner region (figure 1): $z = O(1)$, $r = O(\epsilon)$. Noting further that within this inner region $u = O(1)$, $v = O(\epsilon)$, $\partial/\partial z = O(1)$ and $\partial/\partial r = O(1/\epsilon)$, we immediately conclude from the relevant creeping-flow equations that, again within the inner region,

$$u = z + o(1).$$

Hence, from the continuity equation

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0$$

and (2.1), it follows that

$$v = -\tfrac{1}{2}r + \epsilon^2 A(z)/r,$$

where the function $A(z)$ is to be obtained from the kinematic condition $\mathbf{u}.\mathbf{n} = 0$ at $r = \epsilon R(z)$, with $\mathbf{n}$ the unit inward normal to the surface. In view of (2.2), this condition is equivalent here to $v = \epsilon R'u$ at $r = \epsilon R$, the prime denoting differentiation with respect to z, and therefore

$$v = -\frac{r}{2} + \frac{\epsilon^2 R}{r}\left(\frac{R}{2} + zR'\right). \tag{2.3}$$

Using the above expressions for u and v in the equations of motion, we find that, to a first approximation, the pressure equals that of the undisturbed flow, here set equal to zero. Hence substituting (2.3) in the normal-stress balance, which to this order of approximation is simply

$$2\left(\frac{\partial v}{\partial r}\right)_{r=\epsilon R} + P = \left(\frac{\gamma}{G\mu l}\right)\left(\frac{1}{\epsilon R}\right),$$

P being the unknown constant pressure within the drop divided by $G\mu$ and γ being the interfacial tension, leads immediately to

$$\epsilon = \gamma/G\mu l$$

and

$$zR' - \nu R = -\tfrac{1}{2}, \quad \nu \equiv \tfrac{1}{2}P - 1, \tag{2.4}$$

with boundary conditions $R(\pm 1) = 0$. Clearly, the appropriate solution to (2.4) is

$$R = (2\nu)^{-1}(1 - |z|^{\nu}), \tag{2.5}$$

which is seen to be identical to (1.1) and therefore to Buckmaster's (1972) earlier result.

How ν is to be determined is not an easy question to answer. Buckmaster (1972) proposed setting ν equal to an even integer, thereby rendering the expression (2.5) analytic for all $-1 \leqslant z \leqslant 1$. The possibility of ν not being an even integer cannot, however, be immediately discarded for the following reason. The inner solution, on the basis of which (2.5) was derived, applies only within that region of the flow where $u = O(1)$, $v = O(\epsilon)$, $z = O(1)$ and $r = O(\epsilon)$, i.e. close to the drop. Thus, when z and r, and therefore u and v, are all $O(1)$, a different expansion will in general be required. We shall call this region the outer region (figure 1). In addition, however, in view of (2.1) there exists yet a third region, to be referred to as the singular region (figure 1), where z, r, u and v are all $O(\epsilon)$ and where again the inner solution may no longer hold. To be sure, as will be shown later in this section, the inner solution applies for all $-1 \leqslant z \leqslant 1$ when ν is an even integer and, therefore, the singular region will be absent under these conditions. In general, though, the approximations leading to (2.4) fail when $|z| \leqslant O(\epsilon)$, and it becomes necessary to consider in detail all three regions† just defined and especially the flow in the singular region, which will play a crucial role in eliminating all non-even integer values of ν, i.e. all solutions to (2.4) which are not analytic at the origin.

Let us, then, re-examine the inner solution. In terms of the variables

$$z = z, \quad \bar{r} = r/\epsilon, \quad u = u, \quad \bar{v} = v/\epsilon,$$

the creeping-flow equations expressed in cylindrical co-ordinates become

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\bar{r}\frac{\partial}{\partial \bar{r}}u = \epsilon^2\left\{\frac{\partial p}{\partial z} - \frac{\partial^2 u}{\partial z^2}\right\}, \tag{2.6a}$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\bar{r}\frac{\partial}{\partial \bar{r}}\bar{v} - \frac{1}{\bar{r}^2}\bar{v} - \frac{\partial p}{\partial \bar{r}} = -\epsilon^2\frac{\partial^2 \bar{v}}{\partial z^2}, \tag{2.6b}$$

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{v}) + \frac{\partial u}{\partial z} = 0, \tag{2.6c}$$

with the following boundary conditions at $\bar{r} = R(z)$:

(*a*) the kinematic condition

$$\bar{v} = uR', \tag{2.7a}$$

(*b*) the zero-shear-stress condition

$$\frac{\partial u}{\partial \bar{r}} = -\epsilon^2\left\{\frac{\partial \bar{v}}{\partial z} + 2R'\left(\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\partial u}{\partial z}\right) - R'^2\frac{\partial u}{\partial \bar{r}}\right\} + \epsilon^4(R')^2\frac{\partial \bar{v}}{\partial z}, \tag{2.7b}$$

(*c*) the normal-stress condition

$$P + \frac{1}{1+\epsilon^2 R'^2}\left\{2\frac{\partial \bar{v}}{\partial \bar{r}} - p - 2R'\frac{\partial u}{\partial \bar{r}} - 2\epsilon^2 R'\frac{\partial \bar{v}}{\partial z} + \epsilon^2 R'^2\left(-p + 2\frac{\partial u}{\partial z}\right)\right\}$$
$$= \frac{1}{R\{1+\epsilon^2 R'^2\}^{\frac{1}{2}}}\left\{1 - \frac{\epsilon^2 RR''}{1+\epsilon^2 R'^2}\right\}. \tag{2.7c}$$

In addition, we require that as $\bar{r} \to \infty$ the inner solution matches with the outer solution to be presented shortly.

Equations (2.6), subject to (2.7a, b) and the fact that the outer solution is in terms of the unstretched variable r rather than $\bar{r}$, can be solved by means of a straightforward expansion in ϵ the first few terms of which become

$$p = \epsilon^2\left\{2\frac{d}{dz}(zRR'')\log(\epsilon\bar{r}) + F_1(z)\right\} + \ldots, \tag{2.8a}$$

$$u = z + \epsilon^2\{f(z)\log(\epsilon\bar{r}) + F_2(z)\} + \ldots, \tag{2.8b}$$

† As is the case with the large majority of analyses involving slender bodies, a different expansion is required near the ends of the bubble, $z = \pm 1$. As shown by Buckmaster (1972), however, the indeterminancy in ν cannot be resolved by considering the structure of the flow in this fourth region, which, being exponentially small, will not affect our solution to the order of accuracy to which it will be developed, i.e. to $O(\epsilon^2)$.

$$\bar{v} = -\frac{\bar{r}}{2}+\frac{R}{\bar{r}}\left(\frac{R}{2}+zR'\right)+\epsilon^2\left\{-\frac{\bar{r}}{2}f'\log(\epsilon\bar{r})\right.$$
$$\left.+\frac{R\log(\epsilon R)}{\bar{r}}[R'f+\tfrac{1}{2}Rf']+(\tfrac{1}{4}f'-\tfrac{1}{2}F_2')\left(\bar{r}-\frac{R^2}{\bar{r}}\right)+\frac{RR'}{\bar{r}}F_2\right\}+\ldots, \quad (2.8c)$$

where $$f(z) = 2RR'+zR'^2-zRR''.$$

The functions $F_1(z)$ and $F_2(z)$ are, however, unknown at this stage and must be obtained by matching the above with the appropriate terms of the outer solution. At any rate, to this order of approximation, the normal-stress balance (2.7*c*) becomes

$$zR'-\nu R = -\tfrac{1}{2}-\epsilon^2\{\log(\epsilon R)[5RR'^2+2R^2R''+zR'^3+zRR'R'']$$
$$+2R'^2+zR'^3-\tfrac{1}{4}R'^2-\tfrac{1}{2}RR''+\frac{F_1R}{2}+\frac{d}{dz}(F_2R)\}, \quad (2.9)$$

subject to the condition $R(\pm 1) = 0$.

We now turn to the problem of obtaining a solution in the outer region.

2.2. *The flow in the outer region*

Within the outer region $r = O(1)$, the effect of the slender drop on the flow is equivalent to that produced by a line distribution of singularities along the portion of the z axis within the drop (Batchelor 1970). To first order, only force and mass singularities, i.e. Stokeslets and sources, are needed† and hence the representation of the flow field reduces to that given by Buckmaster (1972), viz.

$$p = \epsilon^2\int_{-1}^{1}\frac{z-t}{s^3}\phi(t)\,dt, \quad s^2 \equiv (z-t)^2+r^2, \quad (2.10a)$$

$$u = z+\epsilon^2\int_{-1}^{1}\left\{\tfrac{1}{2}\phi(t)\left[\frac{1}{s}+\frac{(z-t)^2}{s^3}\right]+g(t)\frac{z-t}{s^3}\right\}dt, \quad (2.10b)$$

$$v = -\frac{r}{2}+\epsilon^2 r\int_{-1}^{1}\left\{\tfrac{1}{2}\phi(t)\frac{z-t}{s^3}+g(t)\frac{1}{s^3}\right\}dt, \quad (2.10c)$$

where $\phi(t)$ and $g(t)$ are, respectively, the unknown Stokeslet and source strengths and satisfy the conditions

$$\int_{-1}^{1}\phi(t)\,dt = 0, \qquad \int_{-1}^{1}g(t)\,dt = 0.$$

These functions can be evaluated by matching (2.10) as $r\to 0$ with the corresponding limits of (2.8) as $\bar{r}\to\infty$. Thus, on making use of the asymptotic forms of the integrals in (2.10) as $r\to 0$ (Tillett 1970; Buckmaster 1972), we find that for $O(\epsilon) < z < 1$

$$\phi(z) = zRR'', \quad g(z) = \tfrac{1}{4}d(zR^2)/dz, \quad (2.11a, b)$$

$$F_1(z) = \phi(z)\frac{2z}{1-z^2}+\phi'(z)\{2-\log[4(1-z^2)]\}+\int_{-1}^{1}\frac{z-t}{|z-t|^3}[\phi(t)-\phi(z)-\phi'(z)(t-z)]\,dt, \quad (2.11c)$$

$$F_2(z) = \phi(z)\{\log[4(1-z^2)]-1\}+\int_{-1}^{1}\frac{\phi(t)-\phi(z)}{|t-z|}dt+g(z)\frac{2z}{1-z^2}$$
$$+g'(z)\{2-\log[4(1-z^2)]\}+\int_{-1}^{1}\frac{z-t}{|z-t|^3}[g(t)-g(z)-g'(z)(t-z)]\,dt. \quad (2.11d)$$

† This can be demonstrated by considering an expansion in the small parameter ϵ of the *exact* integral expressions for $\mathbf{u}$ and p in terms of the axisymmetric force and velocity distributions at the interface.

Then, if we suppose that, to a first approximation (2.11a, b) apply even within the inner region $0 \leqslant |z| \leqslant O(\epsilon)$, F_1 and F_2 are given in terms of R and z, so that by means of an expansion in ϵ we can solve (2.9), which is equivalent to Buckmaster's equation (3.11), and thereby obtain, for any choice of ν, $R(z)$ to $O(\epsilon^2)$ with (2.5) as the leading term. We therefore conclude that, as expected, the inclusion of higher-order terms in the solution for the flow field in the inner and outer regions will not resolve the indeterminacy in ν if the possible existence of the singular region is not taken into account.

2.3. *The flow in the singular region*

We now derive the solution in the singular region and show that this analysis provides the information necessary to determine ν. To do this we must retain in (2.9) terms such as $\epsilon^2 R''$, which will be as significant as $O(1)$ terms when z is $O(\epsilon)$. The leading-order solution in the singular region can be written down immediately by noting that when z is $O(\epsilon)$ the drop is, to a first approximation, an infinite circular cylinder of radius $\epsilon/2\nu$. The flow within this region is then given, again to a first approximation, by

$$p = 0, \quad u = z, \quad v = -\tfrac{1}{2}r + \epsilon^2/8\nu^2 r, \tag{2.12}$$

which is easily seen to satisfy the kinematic, zero-shear-stress and normal-stress conditions at $r = \epsilon/2\nu$ with P, the unknown constant pressure within the drop, set equal to $2(\nu+1)$ as before. Moreover, the above matches with the inner solution.

To obtain the next term in this expansion, we need to examine the expression for $R(z)$ given by the inner solution when $z \to 0$. As shown in appendix A [cf. (A 4)], the latter gives for the shape of the drop when ν is not an even integer

$$r/\epsilon \equiv R(z) = (2\nu)^{-1} + R_n(z) + \epsilon^2 R_a(z),$$

where $R_a(z)$ is an even power series in z and

$$R_n(z) = -\frac{1}{2\nu}|z|^\nu + \epsilon^2 \frac{\nu-1}{8\nu^2}\left\{|z|^{\nu-2}\log\frac{2\nu z}{\epsilon} + \alpha_1 |z|^{\nu-2} + O(|z|^{\nu-4}\log|z|)\right\},$$

where the coefficient α_1 is defined in (A 5). Clearly, the expression for R determined from the solution within the singular region must match with the above in the region of overlap. We further note, however, that to a first approximation the drop within the singular region is a cylinder of radius $\epsilon/2\nu$, and that higher-order terms in R will be obtained by linearization about this cylindrical shape. Consequently, the approximate terms in the solution which will match with R_n and R_a can be determined separately, at least to the next approximation. Now since R_a is analytic, it can be continued to $z = 0$ without any difficulty; on the other hand, R_n is not analytic at $z = 0$ (for ν is not an even integer) and must be matched with the corresponding solution that applies within the singular region. Therefore we need to construct only a solution within the singular region which matches with R_n.

To this end, we find it convenient to introduce the variables

$$(\tilde{z}, \tilde{r}, \tilde{p}, \tilde{u}, \tilde{v}) \equiv (2\nu/\epsilon)(z, r, p, u, v) \tag{2.13}$$

and to represent the surface of the drop by means of

$$(2\nu/\epsilon) r = 1 + \epsilon^\nu \mathscr{R}(\tilde{z}) + 2\nu\epsilon^2 R_a(z), \tag{2.14}$$

where we require that as $|\tilde{z}| \to \infty$

$$\mathscr{R}(\tilde{z}) \to -(|\tilde{z}|/2\nu)^{\nu}. \tag{2.15}$$

Then, in view of (2.12) and (2.14) and if we neglect, for the reasons stated above, the contribution arising from the term $\epsilon^2 R_a(z)$ in (2.14), we have for the pressure and for the velocity components

$$\tilde{p} = \epsilon^{\nu}\tilde{p}^{(1)}(\tilde{z},\tilde{r})+\ldots, \quad \tilde{u} = \tilde{z}+\epsilon^{\nu}u^{(1)}(\tilde{z},\tilde{r})+\ldots, \quad \tilde{v} = -\tfrac{1}{2}(\tilde{r}-1/\tilde{r})+\epsilon^{\nu}v^{(1)}(\tilde{z},\tilde{r})+\ldots,$$

where $p^{(1)}$, $u^{(1)}$ and $v^{(1)}$ satisfy the full creeping-flow equations (since all the terms are of the same order of magnitude) plus the following boundary conditions at $\tilde{r} = 1$:

(*a*) the kinematic condition

$$v^{(1)} = d(\tilde{z}\mathscr{R})/d\tilde{z}, \tag{2.16a}$$

(*b*) the zero-shear-stress condition

$$\frac{\partial u^{(1)}}{\partial \tilde{r}}+\frac{\partial v^{(1)}}{\partial \tilde{z}} = 4\frac{d\mathscr{R}}{d\tilde{z}}, \tag{2.16b}$$

(*c*) the normal-stress balance

$$\frac{d^2\mathscr{R}}{d\tilde{z}^2}+\left(1+\frac{1}{\nu}\right)\mathscr{R}+\frac{1}{2\nu}\left(2\frac{\partial v^{(1)}}{\partial \tilde{r}}-p^{(1)}+P^{(1)}\right) = 0, \tag{2.16c}$$

with $P \equiv 2(\nu+1)+\epsilon^{\nu}P^{(1)}$, where P is once again the as yet undetermined constant pressure within the drop. Also, we require that $u^{(1)}$ be odd in $\tilde{z}$ and that $p^{(1)}$ and $v^{(1)}$ be even, and that they all match with the corresponding terms of the inner and outer solutions in their respective domains of overlap.

The solution of the above can be developed most conveniently by the method of Fourier transforms using the concept of generalized functions as described by Lighthill (1958). We find that

$$p^{(1)} = \int_0^\infty \mathscr{A}(\omega)\,K_0(\omega\tilde{r})\cos(\omega\tilde{z})\,d\omega+p_0, \tag{2.17a}$$

$$u^{(1)} = \int_0^\infty \{\mathscr{B}(\omega)\,K_0(\omega\tilde{r})+\tfrac{1}{2}\mathscr{A}(\omega)\,\tilde{r}K_1(\omega\tilde{r})\}\sin(\omega\tilde{z})\,d\omega, \tag{2.17b}$$

$$v^{(1)} = \int_0^\infty \{[\mathscr{B}(\omega)+\omega^{-1}\mathscr{A}(\omega)]\,K_1(\omega\tilde{r})+\tfrac{1}{2}\mathscr{A}(\omega)\,\tilde{r}K_0(\omega\tilde{r})\}\cos(\omega\tilde{z})\,d\omega \tag{2.17c}$$

which satisfy the creeping-flow equations plus the symmetry conditions about $\tilde{z} = 0$ as well as the requirement that they do not increase exponentially as $\tilde{r} \to \infty$. Here p_0 is an unknown constant pressure which we shall incorporate into $P^{(1)}$, K_0 and K_1 are the modified Bessel functions of the second kind of order zero and one, respectively, and $\mathscr{A}(\omega)$ and $\mathscr{B}(\omega)$ are functions to be determined from the boundary conditions (2.16*a*, *b*). When the resulting expressions are substituted into the normal-stress balance (2.16*c*) we obtain

$$\frac{d^2\mathscr{R}}{d\tilde{z}^2}+\left(1+\frac{1}{\nu}\right)\mathscr{R} = -\frac{P^{(1)}}{2\nu}-\frac{2}{\pi\nu}\int_0^\infty dt\left\{\mathscr{R}(t)\,[F_0(t+\tilde{z})+F_0(t-\tilde{z})]\right.$$
$$\left.+\frac{1}{2}\left[t\frac{d\mathscr{R}}{dt}-\mathscr{R}(t)\right][F_1(t+\tilde{z})+F_1(t-\tilde{z})]\right\}, \tag{2.18a}$$

where

$$F_0(t) = \int_0^\infty d\omega\cos(\omega t)\,\frac{\omega K_1'(\omega)}{K_1(\omega)}, \tag{2.18b}$$

$$F_1(t) = \int_0^\infty d\omega\cos(\omega t)\left[\left(\frac{\omega K_0(\omega)}{K_1(\omega)}\right)^2-1-\omega^2\right]. \tag{2.18c}$$

The above can be solved implicitly under the condition that $\mathscr{R}$ be even in $\tilde{z}$ to yield (writing Λ^2 for $1+1/\nu$)

$$\mathscr{R} = \mathscr{C}\cos(\Lambda\tilde{z}) - \frac{P^{(1)}}{2(\nu+1)} + \frac{4}{\pi\nu}\int_0^\infty\int_0^\infty d\omega\, dt\, \frac{\cos(\omega\tilde{z})\cos(\omega t)}{\omega^2-\Lambda^2}$$
$$\times\left\{\mathscr{R}(t)\frac{\omega K_1'(\omega)}{K_1(\omega)} + \frac{1}{2}\left(t\frac{d\mathscr{R}(t)}{dt} - \mathscr{R}(t)\right)\left[\left(\frac{\omega K_0}{K_1}\right)^2 - 1 - \omega^2\right]\right\}, \quad (2.18d)$$

where $\mathscr{C}$ is an undetermined coefficient and the principal value of the integral is taken. We need to examine now whether (2.18d) can match as $\tilde{z}\to\infty$ with the form of the corresponding perturbation solution of (2.9) as $z\to 0$. To begin with we note immediately that the latter cannot contain trigonometric functions. Therefore, since on application of Lighthill's (1958, p. 51) theorem concerning the asymptotic expression of a Fourier transform we can show that the integral in (2.18) will not have a term $O(\cos\Lambda\tilde{z})$ as $\tilde{z}\to\infty$, we conclude that the coefficient $\mathscr{C}$ in (2.18) must be zero. Also, the constant term $-P^{(1)}/2(\nu+1)$ can be eliminated from (2.18) by letting

$$\mathscr{R}_1(\tilde{z}) = -(2\nu)^\nu\{\mathscr{R}(\tilde{z}) + P^{(1)}/2\nu\}, \quad (2.19)$$

where use has been made of Lighthill's (1958) table 1 in evaluating the double integral of (2.18) when $\mathscr{R}$ is constant, specifically the expression

$$\int_0^\infty \cos(\omega t)\, dt = \pi\delta(\omega). \quad (2.20)$$

Thus the function $\mathscr{R}_1$ satisfies the integral equation

$$\mathscr{R}_1 = \frac{4}{\pi\nu}\int_0^\infty\int_0^\infty d\omega\, dt\,[\cos(\omega\tilde{z})\cos(\omega t)/(\omega^2-\Lambda^2)]$$
$$\times\left\{\mathscr{R}_1(t)\frac{\omega K_1'(\omega)}{K_1(\omega)} + \frac{1}{2}\left(t\frac{d\mathscr{R}_1(t)}{dt} - \mathscr{R}_1(t)\right)\left[\left(\frac{\omega K_0}{K_1}\right)^2 - 1 - \omega^2\right]\right\}, \quad (2.21)$$

which does not have any simple analytic solution. We can, however, study the asymptotic behaviour of the solution for large $\tilde{z}$ by using the techniques presented by Lighthill (1958) and the fact that our solution must satisfy (2.15). We therefore assume that $\mathscr{R}_1 = |\tilde{z}|^\nu$ as $|\tilde{z}|\to\infty$ and then compute from (2.21) the asymptotic form of $\mathscr{R}_1(\tilde{z})$, which must match with the limit of the solution of (2.9). It is shown in appendix A that, for $\nu \neq 2, 4, \ldots$, such an asymptotic analysis of (2.21) yields only identities to at least, $O(|\tilde{z}|^{\nu-2})$, i.e., when the expression

$$\mathscr{R}_1 = |\tilde{z}|^\nu + c_1|\tilde{z}|^{\nu-2}\log|\tilde{z}| + c_2|\tilde{z}|^{\nu-2} + \tilde{\mathscr{R}}, \quad (2.22)$$

where $\tilde{\mathscr{R}}$ is $o(\tilde{z}^{\nu-2})$, is substituted into the integral equation (2.21), the coefficients c_1 and c_2 can be evaluated in terms of ν and are found to be identical with the corresponding coefficients of the solution of (2.9) as $z\to 0$. A closer look at (2.21) and (2.22) reveals, however, that $\tilde{\mathscr{R}}$ is at most $O(|\tilde{z}|^{\nu-4}(\log\tilde{z})^2)$, so that if $1<\nu<3$, $\nu\neq 2$, the Fourier transform of $\tilde{\mathscr{R}}$ exists in the ordinary sense. In fact, on substituting (2.22) into (2.21) and then taking the cosine transform of the resulting expression and using Lighthill's (1958) table 1 where necessary, we obtain (cf. Appendix A) a first-order ordinary differential equation for $\hat{\mathscr{R}}$, the Fourier cosine transform of $\tilde{\mathscr{R}}$:

$$\tfrac{1}{2}\omega\mathscr{K}\, d\hat{\mathscr{R}}/d\omega + \mathscr{L}\hat{\mathscr{R}} = -\sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\mathscr{S}, \quad (2.23)$$

where $$\Gamma(\nu+1) = \int_0^\infty t^\nu e^{-t}\,dt,$$

$$\mathscr{K}(\omega) = (\omega K_0(\omega)/K_1(\omega))^2 - 1 - \omega^2,$$

$$\mathscr{L}(\omega) = \mathscr{K} - \mathscr{H} + \tfrac{1}{2}\nu(\omega^2 - \Lambda^2),$$

$$\mathscr{S}(\omega) = \omega^{-1-\nu}\{\mathscr{H} + \tfrac{1}{2}(\nu-1)\,\mathscr{K} + \tfrac{1}{2}\omega^2\mathscr{K} + [\mathscr{H} + \tfrac{1}{2}(\nu-3)\,\mathscr{K}]\,\omega^2\,\mathscr{M}$$
$$+ (1-\omega^2/\Lambda^2)\,\tfrac{1}{2}(\nu+1)\,(1+\omega^2\mathscr{M})\},$$

$$\mathscr{H}(\omega) = \omega K_1'(\omega)/K_1(\omega)$$

and $$\mathscr{M}(\omega) = \log(2/\omega) - 0{\cdot}577216 + \nu.$$

This differential equation has a formal solution

$$\mathscr{R} = \sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\exp\left[-\int^\omega \frac{2\mathscr{L}}{\omega\mathscr{K}}\,d\omega\right]$$
$$\times\left\{\text{constant} + \int_\omega^\infty \frac{2}{x\mathscr{K}(x)}\exp\left[\int^x \frac{2\mathscr{L}}{t\mathscr{K}}\,dt\right]\mathscr{S}(x)\,dx\right\}. \quad (2.24)$$

The homogeneous solution becomes $O(e^{\nu\omega})$ for large ω and must be eliminated since $\hat{\mathscr{R}}$ must vanish as $\omega\to\infty$ since as $\tilde{z}\to 0$, $\tilde{\mathscr{R}}$ is either $O(1)$, if $\nu > 2$, or at most $O(\tilde{z}^{\nu-2})$, if $\nu < 2$. On the other hand, as $\omega\to 0$ the particular solution has the limiting form

$$\hat{\mathscr{R}} = 2\sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\,\omega^{-(\nu+1)}q(\nu), \quad (2.25)$$

with $$q(\nu) \equiv \int_0^\infty \frac{x^\nu}{\mathscr{K}}\exp\left[\int_0^x \left\{\frac{2\mathscr{L}}{t\mathscr{K}} - \frac{\nu+1}{t}\right\}dt\right]\mathscr{S}(x)\,dx. \quad (2.26)$$

The integral $q(\nu)$ was evaluated numerically for $1 \leqslant \nu \leqslant 3$ and was found to be everywhere positive, thereby implying that $\hat{\mathscr{R}}$ is non-integrable unless, of course, $\nu = 2$. Thus no values of ν are permissible in the range $1 < \nu < 3$ except for $\nu = 2$.

The asymptotic form of $\mathscr{R}_1$ for $\nu = 1$ can also be obtained by taking the appropriate limits of (2.22) and (A 11), and is

$$\mathscr{R}_1(t) \sim |t| - 1/|t| + \tilde{\mathscr{R}},$$

where $\tilde{\mathscr{R}} = o(1/|t|)$. Again, however, the expansion given for $\tilde{\mathscr{R}}$ in (2.25) is not integrable, and hence $\nu \neq 1$. It should be noted though that the analysis developed above does not apply when $\nu = 2$, as can easily be seen from (2.25) and the expression for c_2 in (A 11). This case will be considered separately in detail.

The possibility that $0 < \nu < 1$ can similarly be excluded by repeating the steps shown above. Here, however, the analysis simplifies because it involves, in lieu of $\hat{\mathscr{R}}$, the Fourier transform of $(\mathscr{R}_1 - |\tilde{z}|^\nu)$, denoted here by $\bar{\mathscr{R}}$, which exists in the ordinary sense and can be shown to satisfy

$$\tfrac{1}{2}\omega\mathscr{K}d\bar{\mathscr{R}}/d\omega + \mathscr{L}\bar{\mathscr{R}} = -\sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\,\bar{\mathscr{S}},$$

where $$\bar{\mathscr{S}} = \omega^{-1-\nu}\{\mathscr{H} + \tfrac{1}{2}(\nu-1)\,\mathscr{K} + \tfrac{1}{2}(\nu+1)\}.$$

The solution to the above which vanishes for $\omega\to\infty$ is

$$\bar{\mathscr{R}} = \sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\int_\omega^\infty \frac{2\bar{\mathscr{S}}}{x\mathscr{K}(x)}\exp\left(\int_\omega^x \frac{2\mathscr{L}}{t\mathscr{K}}\,dt\right)dx,$$

whose limiting form as $\omega \to 0$ becomes

$$\bar{\mathscr{R}} = 2\sin(\tfrac{1}{2}\nu\pi)\,\Gamma(\nu+1)\,\omega^{-(\nu+1)}\bar{q}(\nu),$$

with
$$\bar{q}(\nu) \equiv \int_0^\infty \frac{x^\nu}{\mathscr{K}(x)}\exp\left[\int_0^x \left\{\frac{2\mathscr{L}}{t\mathscr{K}} - \frac{\nu+1}{t}\right\} dt\right] \bar{\mathscr{F}}(x)\,dx > 0.$$

Again, though, this is an unacceptable result because it implies that $\bar{\mathscr{R}}$ is non-integrable, contrary to what is known from (2.22), the asymptotic form of $\mathscr{R}_1$ as $|\tilde{z}| \to \infty$, which matches with the corresponding solution of the inner equation (2.9) as $z \to 0$.

Similarly, if $3 < \nu < 5$, it is not difficult to see that the remainder left upon subtracting the $O(|\tilde{z}|^{\nu-4}(\log\tilde{z})^2)$, $|\tilde{z}|^{\nu-4}\log\tilde{z}$ and $|\tilde{z}|^{\nu-4}$ terms from $\bar{\mathscr{R}}$ is well behaved for large $\tilde{z}$ only if $\nu = 4$, and likewise for $\nu = 6$, 8, etc. Such an analysis is not required, however, because, as we shall see in the next section, all solutions to (2.4) with $\nu > 2$ are unstable and therefore of no physical interest.

On the other hand when $\nu = 2$ (similar results also hold for $\nu = 4$, 6, 8, etc.) the limiting forms of the outer solution (2.10) for small r near the mid-plane $z = 0$ are the same as the limiting forms of the inner solution (2.8) for small z. Substituting (2.5) with $\nu = 2$ into (2.10*a*) and (2.11) gives

$$\epsilon^{-2}p = -\frac{1}{8}\int_{-1}^{1} dt(t-t^3)\,(z-t)\,[(z-t)^2+r^2]^{-\frac{3}{2}},$$

so

$$\lim_{r\to 0}\{\lim_{z\to 0}\epsilon^{-2}p\} = \lim_{r\to 0}\{-\tfrac{1}{8}[3(1+r^2)^{\frac{1}{2}} - (2+3r^2)\log(r^{-1}+r^{-1}(1+r^2)^{\frac{1}{2}})]\} = -\tfrac{1}{4}(\log\tfrac{1}{2}r+\tfrac{3}{2}).$$

On the other hand, from the inner solution (2.8*a*),

$$\lim_{r\to 0}\epsilon^{-2}p = \tfrac{1}{8}(1-3z^2)\{-2\log\tfrac{1}{2}\epsilon\bar{r}+\log(1-z^2)\} - \frac{3}{8}\int_{-1}^{1}\frac{t^2-z^2}{|t-z|}\,dt,$$

$$\lim_{z\to 0}\{\lim_{r\to 0}\epsilon^{-2}p\} = -\tfrac{1}{4}(\log\tfrac{1}{2}\epsilon\bar{r}+\tfrac{3}{2}).$$

By similar methods we obtain

$$\lim_{r\to 0}\{\lim_{z\to 0}(p,u,v)\} = \lim_{z\to 0}\{\lim_{r\to 0}(p,u,v)\} \tag{2.27}$$

and that the normal-stress balance (2.9) also applies for all $|z| \geqslant 0$. Therefore (2.5) with $\nu = 2$ is uniformly valid for all $0 \leqslant |z| < 1$. On solving (2.9) with $\nu = 2$ we obtain for the shape of the drop to $O(\epsilon^2)$

$$\begin{aligned} r/\epsilon \equiv R = {} & \tfrac{1}{4}(1-z^2)+\tfrac{1}{8}\epsilon^2\{[\tfrac{7}{4}\log\epsilon+\tfrac{1}{8}(23-42\log 2)] \\ & + z^2[-4\log\epsilon - \tfrac{1}{4}(23-48\log 2) - \tfrac{1}{6}\pi^2] + z^4[\tfrac{9}{4}\log\epsilon+\tfrac{1}{8}(23-54\log 2)] \\ & - (\tfrac{1}{8}+z^2-\tfrac{9}{8}z^4)\log(1-z^2) + \sum_1^\infty n^{-2}z^{2n+2}\}. \end{aligned} \tag{2.28}$$

Note that to $O(\epsilon^2)$

$$R(0) = \tfrac{1}{4}+\tfrac{7}{32}\epsilon^2\log\epsilon+\tfrac{1}{64}(23-42\log 2)\,\epsilon^2,$$

and hence that the asymptotic value $R(0) = \frac{1}{4}$ is approached from below. Comparison of the above with figure 4 of Youngren & Acrivos (1976) indicates, therefore, that the numerical results reported by these authors are somewhat inaccurate at large drop deformations.

We have thus shown that, when $0 < \nu < 3$ but $\nu \neq 2$, a singular region $z = O(\epsilon)$ exists within which the solution to the appropriate flow equations is not well behaved in the sense that it cannot match with the corresponding inner solution in the domain of overlap of the two regions, and have anticipated a similar result for all ν not equal to an even integer. On the other hand, when ν is an even integer the inner solution (2.9) is valid for all $0 \leqslant |z| < 1$ and the singular region is no longer present; hence we conclude that the only permissible choices for ν are even integers. We have therefore recovered Buckmaster's (1972) result but the proof is much stronger.

To a first approximation then, the shape of the drop is

$$r = (\epsilon/2\nu)(1-z^{\nu}), \quad \nu = 2, 4, 6, 8, \ldots, \quad \epsilon \equiv \gamma/G\mu l, \tag{2.29}$$

from which the deformation relation

$$l/a = \beta(\nu)(G\mu a/\gamma)^2, \tag{2.30}$$

where
$$\beta(\nu) = \tfrac{4}{3}(\nu+1)(2\nu+1), \tag{2.31}$$

easily follows, a being the radius of the sphere with the same volume as the drop. It is evident from (2.30) that, according to a steady-state analysis based on the creeping-flow equations, an inviscid drop will extend indefinitely without breaking up when the shear rate is increased. As will be seen in §§ 3 and 4, however, this is no longer the case when the viscosity of the fluid in the drop is finite and when inertial effects are taken into account.

2.4. *The stability of the solution*†

We have just shown that, to a first approximation, the shape of the bubble is given by (2.5) with ν a positive even integer, although we have not completely eliminated the possibility that ν could take any value greater than or equal to 3. We shall now prove that all these shapes are unstable except for that corresponding to $\nu = 2$.

Our analysis will again be based on the creeping-flow equations, with the inertial terms set identically equal to zero. The time-dependent term will, however, be retained in the kinematic condition and thus the system of equations and boundary conditions will simulate the time variation in the shape from some given initial state.

We begin by noting that it is no longer convenient to retain l, the half-length of the bubble, as the characteristic length since in our time-dependent problem this quantity varies with time. A better alternative is to render all radial distances dimensionless with $\gamma/G\mu$, all axial distances with $a(G\mu a/\gamma)^2$, the time with $1/G$, and to denote the equation of the surface by $r = R^*(z,t)$ rather than by $\epsilon R(z)$. Since a is, by definition, the radius of the equivalent sphere, the constant-volume requirement becomes, in these new dimensionless variables,

$$\int_{-l^*}^{l^*} R^{*2}\,dz = \tfrac{4}{3}, \quad \text{or} \quad \int_{-l^*}^{l^*} R^*\frac{\partial R^*}{\partial t}\,dz = 0, \tag{2.32}$$

where t is the dimensionless time and an asterisk denotes a time-dependent quantity. Also, the kinematic condition is

$$v = z\frac{\partial R^*}{\partial z} + \frac{\partial R^*}{\partial t} \quad \text{at} \quad r = R^*(z,t). \tag{2.33}$$

† This analysis was suggested and partly developed by Dr E. J. Hinch of Cambridge University, to whom the authors are grateful.

By repeating the steps in § 2.1 leading to (2.4), it is not difficult to see that the equation for the shape of the bubble is

$$\frac{\partial R^*}{\partial t}+z\frac{\partial R^*}{\partial z}-\nu^* R^* = -\tfrac{1}{2}, \quad R^*(\pm l^*, t) = 0, \tag{2.34}$$

where
$$\nu^*(t) \equiv \tfrac{1}{2}P-1 = -\frac{1}{2}+\frac{3}{8}\int_{-l^*}^{l^*} R^* dz, \tag{2.35}$$

the latter expression being obtained on multiplying (2.34) through by R^*, integrating with respect to z from $-l^*$ to l^* and making use of (2.32). We have already shown [cf. (2.5), (2.30) and (2.31)] that a steady-state solution of the above, here denoted by $R(z)$ with ν the corresponding value of ν^*, is given by

$$R(z) = (2\nu)^{-1}[1-|z|^\nu/l^\nu\}, \quad l = \tfrac{4}{3}(\nu+1)(2\nu+1), \tag{2.36}$$

where $\nu = 2$ or $\nu \geqslant 3$. On then letting

$$R^* = R+R', \quad \nu^* = \nu+\nu',$$

where the primes here denote small quantities, we obtain the small disturbance form of (2.34) and (2.35):

$$\frac{\partial R'}{\partial t}+z\frac{\partial R'}{\partial z}-\nu R'-\frac{\nu'}{2\nu}\left\{1-\frac{|z|^\nu}{l^\nu}\right\} = 0, \tag{2.37}$$

where
$$\nu' = \frac{3}{8}\int_{-l}^{l} R'\,dz \sim \frac{3l}{8}\int_{-1}^{1} R'\,dx \quad \text{with} \quad x \equiv z/l. \tag{2.38}$$

Therefore, on making use of the expression for l given by (2.36) and transforming co-ordinates from z to x, we arrive at

$$\frac{\partial R'}{\partial t}+x\frac{\partial R'}{\partial x}-\nu R'-\frac{(\nu+1)(2\nu+1)}{4\nu}\left(\int_{-1}^{1} R'\,dx\right)\{1-|x|^\nu\} = 0, \tag{2.39}$$

together with the condition, on account of (2.32), that

$$\int_{-1}^{1}(1-|x|^\nu)R'\,dx = 0. \tag{2.40}$$

Equation (2.39) can be solved by separation of variables; i.e. we let

$$R' = e^{\sigma t}f(x), \tag{2.41}$$

where, without loss of generality, we set

$$\frac{(\nu+1)(2\nu+1)}{4\nu}\int_{-1}^{1} f(x)\,dx = 1.$$

On substituting (2.41) into (2.39) and solving the resulting ordinary differential equation, we obtain

$$f(x) = C|x|^{\nu-\sigma}+\frac{1}{\sigma-\nu}-\frac{|x|^\nu}{\sigma} \quad \text{if} \quad \sigma \neq \nu \quad \text{and} \quad \sigma \neq 0, \tag{2.42}$$

where, in general, the coefficient C is determined from (2.40), which becomes

$$\int_{-1}^{1}(1-|x|^\nu)f(x)\,dx = 0. \tag{2.43}$$

Equation (2.42) represents the even solution; another solution is given by

$$\frac{1}{\sigma-\nu}-\frac{|x|^\nu}{\sigma}+C_1|x|^{\nu-\sigma}\operatorname{sgn} x, \tag{2.44}$$

where C_1 remains unknown. One eigenvalue σ is obtained readily by setting $C = ($ in (2.42) and determining σ from (2.43). The result is $\sigma = -\frac{1}{2}$ for all ν and the corresponding eigenfunction is

$$f(x) = -(\nu+\tfrac{1}{2})^{-1} + 2|x|^{\nu}. \tag{2.45}$$

It is easy to show, moreover, that this is just the linearized form of the shape-preserving solution to (2.34) and (2.35), subject to (2.32):

$$R^* = R_0(t)\{1-(|x|/l^*)^{\nu}\}, \tag{2.46}$$

where R_0 and $l^*(t)$ satisfy

$$\frac{dR_0}{dt} + \frac{R_0}{2} = \frac{1}{4\nu}, \quad l^* = \frac{(\nu+1)(2\nu+1)}{3\nu^2} R_0^{-2} \tag{2.47}$$

respectively. Thus, for any value of ν, (2.5) is stable to the corresponding shape-preserving disturbance.

On the other hand, to determine the remaining eigenvalues σ, it is again necessary to examine the solution within the singular region already discussed in § 2.3. The analysis is straightforward, and as shown in appendix B, an equation similar to (2.18a) is obtained for $\mathscr{R}'$ (the perturbation in $\mathscr{R}$), which becomes

$$\frac{d^2\mathscr{R}'}{dz^2} + \left(1+\frac{1}{\nu}\right)\mathscr{R}' = -\frac{P^{(1)'}}{2\nu} - \frac{2}{\pi\nu}\int_0^{\infty} d\eta \left\{\mathscr{R}'(\eta,t)\left[F_0(\eta+z)+F_0(\eta-z)\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(-\mathscr{R}'(\eta,t) + \eta\frac{\partial\mathscr{R}'}{\partial\eta} + \frac{\partial\mathscr{R}'}{\partial t}\right)\left[F_1(\eta+z)+F_1(\eta-z)\right]\right\}. \tag{2.48}$$

In fact, the only difference between (2.48) and (2.18a) is in the inhomogeneous term. At any rate, upon separating variables as in (2.41) and retracing the steps in § 2.3 we conclude that:

(a) If $f(x)$ is even, then $\nu-\sigma = 2$ or $\geqslant 3$, with $\sigma \neq \nu$ and $\sigma \neq 0$ [cf. (2.42)]. Therefore the steady solution with $\nu = 2$ is stable, whereas all other steady solutions with $\nu \geqslant 3$ (in fact, all solutions with $\nu > 2$) are unstable because the lowest eigenvalue is then $\nu - 2$, which is positive.

(b) If $f(x)$ is odd [cf. (2.44)], consideration of (2.48) along the lines of § 2.3 for functions $\mathscr{R}$ that are odd shows that $\nu-\sigma = 1$ *is* a possible solution but can be eliminated by a shift in the origin, whose location has been assumed fixed in deriving (2.37).

Thus we conclude that the only choice of ν which leads to a steady bubble shape for $-1 \leqslant z \leqslant 1$ is $\nu = 2$. The problem as originally posed therefore has a unique solution.

3. A viscous drop ($\lambda \neq 0$) in zero-Reynolds-number flow

We next consider the effect of the viscosity of the fluid inside the drop. Inertial effects are again assumed negligible so that the creeping-flow equations apply both inside and outside the drop. This case has already been studied by Buckmaster (1973) and hence we shall only briefly sketch the main steps of the analysis.

The dimensionless field equations for the exterior region are the same as those in § 2.1, while those for the interior region become

$$\frac{\partial p}{\partial z} - \frac{\lambda}{\epsilon^2}\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\bar{r}\frac{\partial}{\partial \bar{r}}u = \lambda\frac{\partial^2 u}{\partial z^2}, \tag{3.1a}$$

$$\frac{\partial p}{\partial \bar{r}} = \lambda\left(\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\bar{r}\frac{\partial}{\partial \bar{r}} - \frac{1}{\bar{r}^2}\right)\bar{v} + \lambda\epsilon^2\frac{\partial^2 \bar{v}}{\partial z^2}, \tag{3.1b}$$

$$\frac{\partial u}{\partial z} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\bar{v}) = 0, \tag{3.1c}$$

where λ is the ratio of the viscosity μ_i of the dispersed phase to that of the suspending phase, μ. Since we already know, however, from the normal-stress balance for slender drops [cf. the expression following (2.3)] that p is an $O(1)$ quantity within the drops, (3.1 a) implies that slender drops can exist only if $\lambda \leqslant O(\epsilon^2)$. Thus we are led to consider the non-trivial case $\lambda = O(\epsilon^2)$, which implies that

$$K^2 \equiv \epsilon^2/\lambda = O(1). \tag{3.2}$$

It immediately follows from (3.1) that to leading order in ϵ the shape R depends only on z and on the dimensionless parameter K^2, i.e. the equation for the drop interface is

$$\bar{r} = R(z; K^2). \tag{3.3}$$

Equating the volume $\frac{4}{3}\pi a^3$ of the equivalent sphere to that of the drop,

$$l^3\pi\int_{-1}^{1}(\epsilon R)^2\,dz,$$

and again letting $\epsilon = \gamma/G\mu l$, we find that

$$\beta \equiv \frac{l}{a}(\gamma/G\mu a)^2 = \frac{2}{3}\left\{\int_0^1 R^2 dz\right\}^{-1} = \text{function of } K^2 \text{ only}. \tag{3.4}$$

From the definition of β, K^2 and ϵ we can then form the dimensionless rate of strain

$$\mathscr{G} \equiv (G\mu a/\gamma)\lambda^{\frac{1}{6}} = (K^2\beta^2)^{-\frac{1}{6}} \tag{3.5}$$

and the dimensionless extension of the drop

$$L \equiv (l/a)\lambda^{\frac{1}{3}} = \beta\mathscr{G}^2. \tag{3.6}$$

Noting that β is a function of K^2 only, we can compute β and therefore $\mathscr{G}$ and L for any specified value of K^2. The deformation curve $L(\mathscr{G})$ can then be readily obtained once R has been determined.

Equations (3.1) for $\lambda = O(\epsilon^2)$ have been solved by Buckmaster (1973), who obtained the expression for the internal pressure field [his equation (13)]:

$$p(z) = p(0) + \frac{8}{K^2}\int_0^z \frac{s\,ds}{R^2(s)}, \tag{3.7}$$

where $p(0)$ is the pressure at $z = 0$ inside the drop, with the pressure of the undisturbed flow field at infinity set equal to zero. Also

$$u = z(2\bar{r}^2/R^2 - 1), \tag{3.8}$$

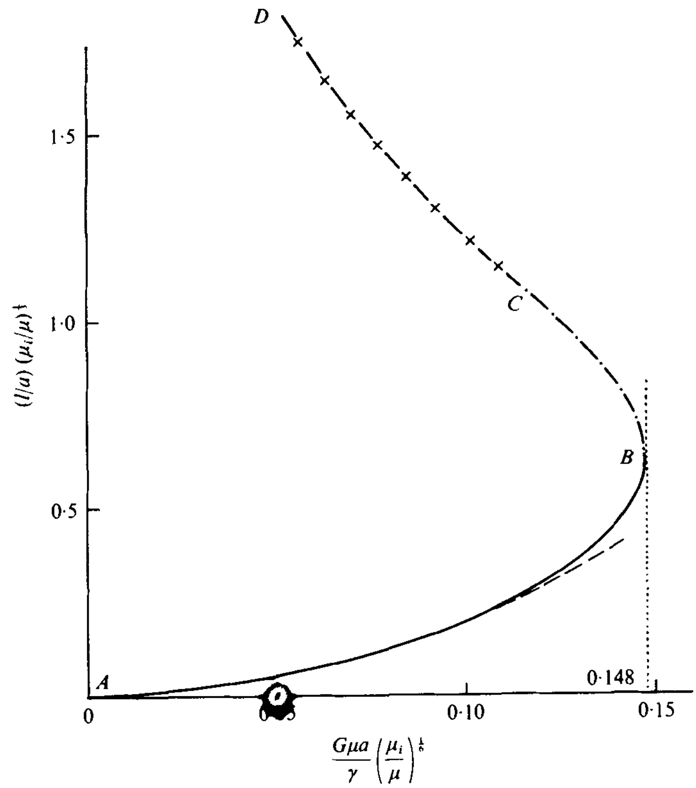

FIGURE 2. The first two branches of the deformation curve for a drop with finite viscosity and zero inertial effects. ——, stable drop deformation curve; – – –, reference curve for zero-viscosity case; –·–·–·, - + - +, unstable steady-state solutions; B, point of breakup; C, point at which the two branches join.

which together with (3.7) is seen to satisfy, to $O(\epsilon^2)$, (3.1 a) plus the boundary conditions of continuity of tangential velocity, $u = z$ at $\bar{r} = R$, continuity of shear stress, $\lambda\, du/d\bar{r} = O(\epsilon^2)$ at $\bar{r} = R$, and mass conservation,

$$\int_0^R u\bar{r}\, d\bar{r} = 0. \tag{3.9}$$

The flow outside the drop is unchanged, to this order or approximation, so that the normal-stress balance is just (2.4) with $p(z)$ as given in (3.7) replacing the constant P. The resulting equation is then

$$zR' - \left(\nu + \frac{4}{K^2}\int_0^z \frac{s\,ds}{R^2}\right) R = -\tfrac{1}{2}, \quad \nu \equiv \tfrac{1}{2}p(0) - 1, \quad R(\pm 1) = 0, \tag{3.10a}$$

or upon differentiation

$$2zRR'' + 2RR' - 2zR'^2 - R' = 8z/K^2, \quad R(0) = (2\nu)^{-1}, \quad R(\pm 1) = 0. \tag{3.10b}$$

Once again, we observe that the system is indeterminate because $p(0)$, or equivalently $R(0)$, is unknown *a priori*. By analogy to the case of an inviscid drop ($\lambda = 0$), though, we can show that the normal-stress balance in a region very close to $z = 0$ is the same as (2.16c), except that $P^{(1)}$ is now a quadratic function of z. The solution of (2.16c) is again very similar to (2.18) plus an analytic function; however, in view of the discussion following (2.12), this analytic portion of the solution can be considered independently of the non-analytic part, which satisfies an equation essentially identical to (2.21). But, as we have shown, the latter does not possess a solution which will match with the solution of (3.10) as $z \to 0$, and hence we conclude that R, as determined from (3.10), must be analytic at $z = 0$.

As with the earlier case $\lambda = 0$, the solution of (3.10) is non-unique even if we require that it be analytic at $z = 0$. On account of the stability analysis in § 2.4, however, we can take it for granted that, of all these possible solutions, only that with the lowest permissible value of ν, i.e. that giving a drop shape with the least deformation, is stable. This solution has the two branches (Taylor 1964; Buckmaster 1973):

$$R(z) = \tfrac{1}{8}[1 + (1 - 64/K^2)^{\frac{1}{2}}](1 - z^2) \tag{3.11a}$$

and

$$R(z) = \tfrac{1}{8}[1 - (1 - 64/K^2)^{\frac{1}{2}}](1 - z^2), \tag{3.11b}$$

which evidently require that $K^2 \geqslant 64$. On substituting the above in (3.4)–(3.6) we obtain the deformation curve [equivalent to Taylor's (1964) equation (16)]

$$\frac{G\mu a}{\gamma}\lambda^{\frac{1}{6}} = \frac{1}{\sqrt{20}}\left(\frac{l}{a}\lambda^{\frac{1}{3}}\right)^{\frac{1}{2}} \bigg/ \left[1 + \frac{4}{5}\left(\frac{l}{a}\lambda^{\frac{1}{3}}\right)^3\right], \tag{3.12}$$

which is plotted in figure 2, the section ABC corresponding to (3.11a) and CD to (3.11b). Along ABC the parameter K^2 decreases monotonically from infinity to 64 then increases monotonically back to infinity along CD.

We shall now prove that the steady-state solutions represented by the upper part of this curve, i.e. the section BCD, are unstable and therefore of no physical significance. Let us consider the time-dependent deformation of the drop. With the inertial terms again set equal to zero everywhere, but with the time derivative retained in the kinematic condition, Buckmaster (1973) has shown that the relevant mathematical system admits the shape-preserving solution

$$R^*(z,t) = R_0(t)\,[1 - z^2/l^{*2}],$$

where $R_0(t)$ satisfies his equation (41), which, on account of the constant-volume requirement

$$R_0^2 l^* = \tfrac{5}{4}a^3,$$

can be recast as

$$\frac{G\mu a}{\gamma}\lambda^{\frac{1}{6}}\frac{d\log l^*}{dt} = \frac{G\mu a}{\gamma}\lambda^{\frac{1}{6}} - \frac{1}{(20)^{\frac{1}{2}}}\left(\frac{l^*}{a}\lambda^{\frac{1}{3}}\right)^{\frac{1}{2}} \bigg/ \left[1 + \frac{4}{5}\left(\frac{l^*}{a}\lambda^{\frac{1}{3}}\right)^3\right], \tag{3.13}$$

t being the time rendered dimensionless with $1/G$. Since, with reference to figure 2 and (3.12), the right-hand side of (3.13) is negative within the region enclosed by the curve $ABCDA$ and positive outside it, it is evident that only the steady-state solutions lying along the section AB are stable to shape-preserving disturbances. Moreover, since the value of K^2 at B is found to equal $\frac{576}{5}$, it follows that the only solution of (3.10) with physical significance is given by (3.11a) with $\frac{576}{5} \leqslant K^2 \leqslant \infty$. Thus Buckmaster's (1973) criterion for breakup ($K^2 = 64$ at C in figure 2) is seen to lie outside the range of stable solutions.

It is tempting at this stage to identify B with the point of drop breakup since, according to our analysis, for dimensionless shear rates exceeding the critical value

$$\mathscr{G}^{+} = \frac{G_c \mu a}{\gamma} \lambda^{\frac{1}{6}} = \tfrac{1}{3} \times 5^{\frac{1}{2}}/2^{\frac{7}{3}} = 0{\cdot}148\ldots \tag{3.14}$$

a steady shape cannot exist. Before reaching this conclusion, however, it is necessary to establish that the solutions lying along the curve AB in figure 2 are stable to all disturbances and not only to those of the shape-preserving type. This can be achieved by means of a stability analysis similar to that in § 2.4. Only axisymmetric disturbances will be examined since these are believed to be the least stable.

Again, it is convenient to employ a non-dimensional set of variables which differs somewhat from that used up to now. As before, we let the characteristic time be $1/G$; however we now choose the characteristic dimension in the radial direction to be equal to $a\lambda^{\frac{1}{6}}$, that in the axial direction to be equal to $a\lambda^{-\frac{1}{3}}$ and, as in § 2.4, denote the equation of the surface by $r = R^*(z,t)$ rather than by ϵR. The equation for the drop shape then becomes

$$R^* \frac{\partial^2 R^*}{\partial z\, \partial t} - \frac{\partial R^*}{\partial z}\frac{\partial R^*}{\partial t} + zR^* \frac{\partial^2 R^*}{\partial z^2} + \frac{\partial R^*}{\partial z} - z\left(\frac{\partial R^*}{\partial z}\right)^2 - \frac{1}{2\mathscr{G}_1}\frac{\partial R^*}{\partial z}$$
$$= 4z + \frac{4}{R^{*2}}\frac{\partial}{\partial t}\int_0^z R^{*2}\, ds, \tag{3.15}$$

which in a steady state can easily be shown to reduce to (3.10*b*) when account is taken of the difference in the notation.

Let
$$R^* = \tfrac{1}{2} \times 5^{\frac{1}{2}} L^{-\frac{1}{2}}(1 - x^2) + R', \tag{3.16}$$
where the first term represents the steady-state solution, R' is a small perturbation from this steady shape and $x \equiv z/L$. On substituting the above in (3.15) we obtain, to first order in R',

$$(1-x^2)\frac{\partial^2 R'}{\partial x\, \partial t} + 2x\frac{\partial R'}{\partial t} + x(1-x^2)\frac{\partial^2 R'}{\partial x^2} - 4xR' + (1+3x^3)\frac{\partial R'}{\partial x}$$
$$- \frac{L^{\frac{1}{2}}}{\mathscr{G} 5^{\frac{1}{2}}}\frac{\partial R'}{\partial x} = \frac{32}{5}\frac{L^3}{(1-x^2)^2}\frac{\partial}{\partial t}\int_0^x (1-s^2)\, R'\, ds, \tag{3.17}$$

where $\mathscr{G}$ and L are related by means of (3.12), i.e.

$$\mathscr{G} = (20)^{-\frac{1}{2}} L^{\frac{1}{2}}/(1 + \tfrac{4}{5}L^3), \tag{3.18}$$

in which only the lower branch, i.e. that represented by the curve AB in figure 2, is of interest.

As in § 2.4 we let $R' = e^{\sigma t} f(x)$, which on using (3.18) transforms (3.17) into

$$x(1-x^2)\frac{d^2 f}{dx^2} + \{\sigma(1-x^2) + 1 + 3x^2 - 2 + \tfrac{8}{5}L^3\}\frac{df}{dx} + \{2x\sigma - 4x\}f$$
$$= \tfrac{32}{5}\,\sigma \frac{L^3}{(1-x^2)^2}\int_0^x (1-s^2) f\, ds. \tag{3.19}$$

Also, the constant-volume requirement becomes

$$\int_0^1 (1-s^2) f\, ds = 0. \tag{3.20}$$

The solution of (3.19), with $f(0)$ set equal to unity without loss of generality, is

$$f = 1 + \frac{2-\sigma+\frac{16}{5}\sigma L^3}{\sigma-\frac{8}{5}L^3}x^2 + C\{x^{2+\frac{8}{5}L^3-\sigma}+\ldots\}, \tag{3.21}$$

if
$$\sigma \neq \tfrac{8}{5}L^3 \quad \text{and} \quad \sigma \neq 2+\tfrac{8}{5}L^3,$$

where C is a constant of integration. With reference to our earlier analysis, however, and especially that in § 2.4, we require that (3.21) be analytic at $x = 0$. This gives the following results:

(*a*) If $C = 0$, the eigenvalue σ is determined by applying (3.20) to (3.21), and leads to

$$\sigma = -\frac{1}{2(1+\frac{4}{5}L^3)}\{1-4L^3\} < 0 \quad \text{for} \quad \mathscr{G} < \mathscr{G}^+,$$

which, as expected, can also be obtained by linearizing (3.13), the equation for the shape-preserving solution.

(*b*) If $C \neq 0$, then, with $n = 4, 6, \ldots,$

$$\sigma = -n+\tfrac{8}{5}L^3+2,$$

all of which are negative since, in view of (3.18),

$$0 \leqslant L^3 \leqslant \tfrac{1}{4} \quad \text{for} \quad 0 \leqslant \mathscr{G} \leqslant \mathscr{G}^+.$$

We conclude therefore that, for $\mathscr{G} < \mathscr{G}^+ < 0{\cdot}148\ldots$, the steady drop shape given by (3.11*a*) is stable to all small axisymmetric disturbances and hence that the critical shear rate for drop breakup is correctly given by (3.14). At this critical value of $\mathscr{G}$, the dimensionless extension L equals $4^{-\frac{1}{3}} \simeq 0{\cdot}63$ and therefore, from (3.16), the slenderness ratio of the drop, i.e. the ratio of its radius at $z = 0$ to its half-length, is equal to $(5\lambda)^{\frac{1}{2}}$. However, since the present analysis remains accurate only if this slenderness ratio is $O(10^{-1})$ or less, we conclude that the criterion developed above for drop breakup should apply only if $\lambda \leqslant O(10^{-2})$.

4. An inviscid drop ($\lambda \neq 0$) in flow at non-zero Reynolds number

It has already been remarked that, in view of the very substantial length that many drops attain prior to breakup, the particle Reynolds number may no longer remain small; hence inertia effects may become important and should be taken into account. In fact we shall now show that, as was the case when the internal viscosity inside the drop was finite, the presence of inertia in the flow outside also facilitates the breakup process.

Denoting by $\alpha \equiv \rho G l^2/\mu$ the Reynolds number based on the half-length of the drop, we notice that the Navier–Stokes equations admit an exact solution

$$p = -\tfrac{1}{2}\alpha(z^2+\tfrac{1}{4}r^2), \quad u = z, \quad v = -\tfrac{1}{2}r, \tag{4.1}$$

which is just the impressed extensional flow. Then, provided that the drop is sufficiently slender, the flow outside remains essentially undisturbed, so that to leading order in $\epsilon \equiv \gamma/G\mu l$ the pressure just outside the drop becomes

$$p = -\tfrac{1}{2}\alpha z^2.$$

The normal-stress balance is therefore, in lieu of (2.4),

$$zR' - (\nu+\tfrac{1}{4}\alpha z^2)R = -\tfrac{1}{2}, \quad R(\pm 1) = 0, \quad \nu = \tfrac{1}{2}P-1, \tag{4.2}$$

where P is again the constant pressure inside the drop. Obviously R depends only on the dimensionless parameter α and z, i.e.

$$r = \epsilon R(z;\, \alpha). \tag{4.3}$$

In terms of a, the radius of the equivalent sphere, we again obtain

$$\beta \equiv \frac{l}{a}(\gamma/G\mu a)^2 = \frac{2}{3}\left\{\int_0^1 R^2\, dz\right\}^{-1}, \tag{4.4}$$

where β is a function of α only. By definition

$$\alpha \equiv \frac{\rho G l^2}{\mu} = \left(\frac{G\mu a}{\gamma}\right)^5 \left(\frac{\rho\gamma a}{\mu^2}\right)\beta^2, \tag{4.5}$$

which leads logically to the definition of the dimensionless rate of strain

$$\mathscr{G}_\alpha \equiv \frac{G\mu a}{\gamma}\left(\frac{\rho a\gamma}{\mu^2}\right)^{\frac{1}{5}} = \left(\frac{\alpha}{\beta^2}\right)^{\frac{1}{5}} \tag{4.6}$$

and the dimensionless extension of the drop

$$L_\alpha \equiv \frac{l}{a}\left(\frac{\rho a\gamma}{\mu^2}\right)^{\frac{2}{5}} = \beta\mathscr{G}_\alpha^2. \tag{4.7}$$

The deformation relation $L_\alpha(\mathscr{G}_\alpha)$ can then easily be obtained once $\beta(\alpha)$ has been computed from the solution of (4.2).

By analogy with the previous two cases we require that R, as determined from (4.2), be analytic at $z = 0$. As shown below, the solution of (4.2) near $z = 0$ can be decomposed into two parts, one of which is analytic at $z = 0$ while the other is protional to z^ν, where $\nu \equiv \frac{1}{2}P - 1$ is, in general, not an integer. The coefficient of the z^ν term can then be made to vanish by a proper choice of ν and, as before, there are many such choices which will render R analytic at $z = 0$. However, in view of our earlier analysis, we shall take it for granted that only the solution corresponding to the lowest value of ν thus obtained is stable and therefore of physical interest.

We begin by expressing the formal solution to (4.2) as

$$R = \tfrac{1}{2}z^\nu \exp\left(\tfrac{1}{8}\alpha z^2\right)\int_z^1 t^{-\nu-1}\exp\left(-\tfrac{1}{8}\alpha t^2\right)dt \tag{4.8a}$$

$$= \frac{1}{2\nu}\left\{1 - \frac{\frac{1}{4}\alpha}{\nu-2}z^2 + \frac{\frac{1}{4}\alpha}{\nu-2}\frac{\frac{1}{4}\alpha}{\nu-4}z^4 - \ldots\right\} + \frac{z^\nu \exp\{-\frac{1}{8}\alpha(1-z^2)\}}{2\nu(\nu-2)}\{2-\nu+\Theta(\nu,\alpha)\}, \tag{4.8b}$$

where $$\Theta = \tfrac{1}{4}\alpha\left[1 + \frac{\frac{1}{4}\alpha}{4-\nu} + \frac{(\frac{1}{4}\alpha)^2}{(4-\nu)(6-\nu)} + \ldots\right].$$

The shape of the drop is therefore given by (4.8a) with $\nu(\alpha)$ equal to any one of the roots of $\Theta = \nu - 2$. It is easy to show that the first two roots lie between 2 and 4, the next two between 6 and 8, etc.

The deformation curves corresponding to the first two roots are presented in figure 3. It is worth noting that a maximum shear rate exists. The first two branches (and also pairs of the higher-order branches) join smoothly to form a composite clover-shaped curve, which for small shear rates can be approximated by the corresponding reference curves (broken lines) for $\lambda = 0$, $\rho = 0$. The first branch doubles back on itself at

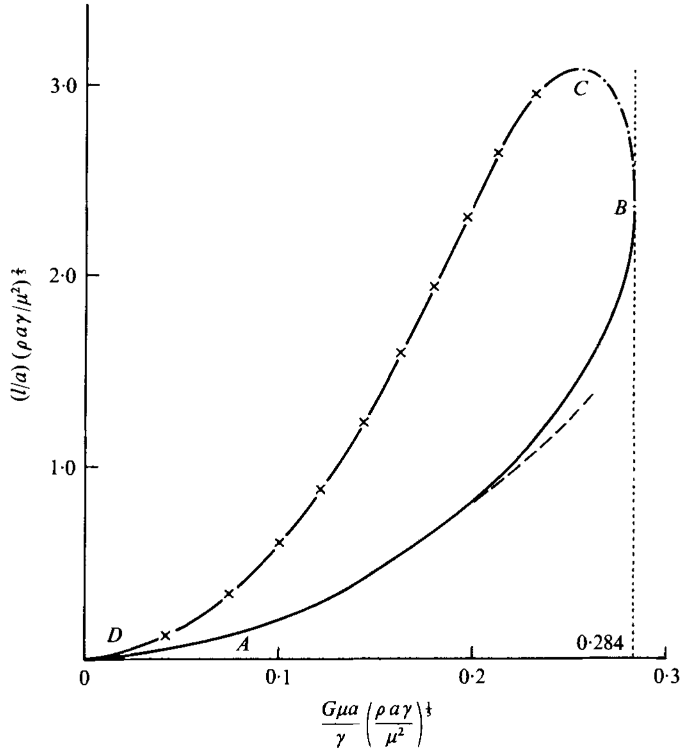

FIGURE 3. The first two branches of the deformation curve for a drop with finite inertial but zero viscosity effects. ——, Stable drop deformation curve; – – –, reference curve for zero-inertia case; –·–·, – + – +, unstable steady-state solutions; B, point of breakup; C, point at which the two branches join.

$\mathscr{G}_\alpha^+ = 0{\cdot}284$, $L_\alpha^+ = 2{\cdot}37$, $\nu^+ = 2{\cdot}52$, which corresponds to the condition at breakup, then stops at the point C at the top of the loop, where it is joined by the second branch. Again, on account of our earlier results, we shall suppose that the lower part of this curve, viz. the section AB, is stable and that the upper, viz the section BCD, is unstable. Also, as there are no steady solutions for $\mathscr{G}_\alpha > \mathscr{G}_\alpha^+$, we conclude that an inviscid drop in a flow at non-zero Reynolds number will break up when $\mathscr{G}_\alpha$ reaches the value $\mathscr{G}_\alpha^+ = 0{\cdot}284$.

At this critical value of $\mathscr{G}_\alpha$, the dimensionless extension L_α is 2·37 and the corresponding slenderness ratio of the drop is $0{\cdot}295(\rho a\gamma/\mu^2)^{\frac{1}{3}}$. However, since the analysis is valid only if the slenderness ratio is $O(10^{-1})$ or less, we conclude that the above results should be valid only if $(\rho a\gamma/\mu^2) \leqslant 0{\cdot}165$, or equivalently, if the Reynolds number based on a, the equivalent radius of the drop, is less than about 0·07. Under these limiting conditions, the Reynolds number based on l, the half-length of the drop, is, approximately, 1·6.

5. An inertialess viscous drop in flow at non-zero Reynolds number

This is in essence a composite of the two cases treated in §§ 3 and 4, and for $\lambda = O(\epsilon^2)$ applies provided the ratio of the density of the fluid within the drop to that of the surrounding medium is sufficiently small, i.e. for a gas bubble in a liquid. The flow

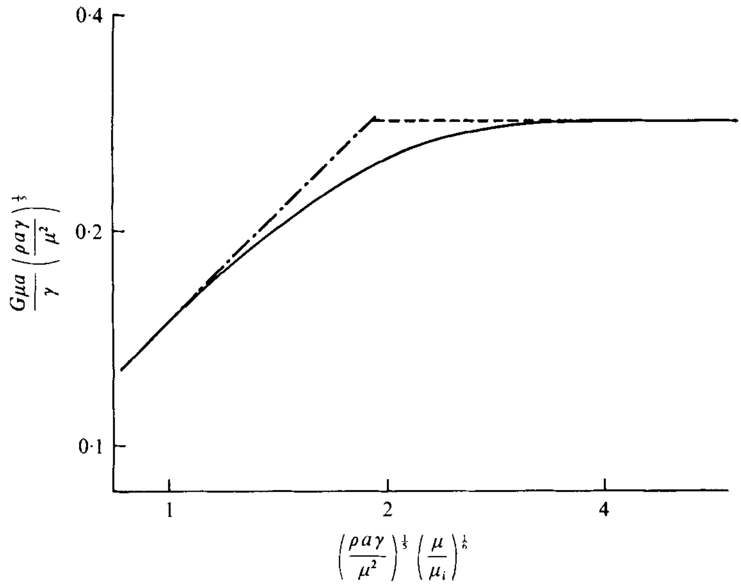

FIGURE 4. The critical shear rate at breakup of a drop having zero density. ——, drop; —·—·, no inertial effects; ---, limit of zero internal viscosity.

outside the drop impresses a pressure $-\frac{1}{2}\alpha z^2$ just outside the interface while the flow inside generates another pressure profile

$$p(0)+\frac{8}{K^2}\int_0^z \frac{s\,ds}{R^2}.$$

The normal-stress balance therefore includes both terms. Thus

$$zR'-\left(\nu+\tfrac{1}{4}\alpha z^2+\frac{4}{K^2}\int_0^z\frac{s\,ds}{R^2}\right)R=-\tfrac{1}{2},\quad R(\pm 1)=0,\quad \nu=\frac{P(0)}{2}-1, \tag{5.1}$$

and hence the shape of the drop is

$$r=\epsilon R(z;\,\alpha,K^2). \tag{5.2}$$

For prescribed values of α and K^2, (5.1) can be solved for functions that are analytic at $z=0$ by considering the behaviour of R near $z=0$ and then obtaining a transcendental equation for $\nu=\nu(\alpha,K^2)$. The two dimensionless strain rates

$$\mathscr{G}\equiv\frac{G\mu a}{\gamma}\lambda^{\frac{1}{6}},\quad \mathscr{G}_\alpha\equiv\frac{G\mu a}{\gamma}\left(\frac{\rho a\gamma}{\mu^2}\right)^{\frac{1}{3}}$$

can then be computed according to (3.7) and (4.6) and their ratio

$$\zeta\equiv\mathscr{G}_\alpha/\mathscr{G}=(\rho a\gamma/\mu^2)^{\frac{1}{3}}\lambda^{-\frac{1}{6}}$$

defines a dimensionless group that depends only on the physical properties of the system. We can then vary α and K^2, which corresponds to the realization of various experiments, and obtain a number of points on a plot of $\mathscr{G}_\alpha(\alpha,K^2)$ *vs.* ζ. The envelope of these points then defines the critical shear rates $\mathscr{G}_\alpha^+(\zeta)$ beyond which drops of any given ζ will break up (figure 4).

In a typical experiment ζ is fixed and the shear rate G is increased gradually until the drop breaks, thereby locating one point on the $\mathscr{G}_\alpha^+(\zeta)$ curve. In a theoretical analysis, however, the form of (5.1), which contains α, K^2 and therefore the *a priori* unknown half-length of the drop l, is such that many different combinations of α and K^2 must be considered before the whole $\mathscr{G}_\alpha^+(\zeta)$ curve can be generated. In view of the excessive amount of computation that an exact analysis would entail, an approximate method was therefore devised which took advantage of the fact that the major contribution of the volume of the drop comes from the region of small z, where $R = R(0) + O(z^2)$. Consequently, the pressure field generated by the internal flow,

$$p(0) + \frac{8}{K^2}\int_0^z \frac{s\,ds}{R^2},$$

was replaced by $p(0) + 16\nu^2 z^2/K^2$, so that the equation for R became identical to that for an inviscid drop at non-zero Reynolds numbers, i.e. (4.2) with α replaced by $\alpha + 32\nu^2/K^2$. For the case of zero inertia this method yielded a critical shear rate which was within 1 % of the exact answer. The approximation should of course improve when α is increased. Figure 4 depicts the results of such an approximate, but accurate, calculation. As expected, the critical shear rate tends to the limits

$$\mathscr{G}_\alpha \to \begin{cases} 0{\cdot}148\zeta & \text{as} \quad \zeta \to 0 \quad (\rho = 0 \text{ case}), \\ 0{\cdot}284 & \text{as} \quad \zeta \to \infty \quad (\lambda = 0 \text{ case}). \end{cases}$$

In fact, accuracy to within 10 % can be achieved if the actual $\mathscr{G}_\alpha^+(\zeta)$ curve is represented by its two asymptotes, which in principle can be obtained by performing only two experiments.

In order to complete this analysis, we should, of course, also consider the situation in which inertial effects are important within the drop as well, i.e. for a liquid drop suspended in another viscous liquid, but unfortunately we have been unable so far to obtain the solution for the corresponding profile owing to the complicated nature of the flow within the drop when the internal Reynolds number is large. Hence the breakup criterion for this case remains to be determined.

6. Generalizations to other flow fields

In principle, the analysis developed so far could be extended to the case of a drop freely suspended in a non-axisymmetric linear flow, in particular a hyperbolic or a simple shear flow, for which, as mentioned in the introduction, experimental results are available when inertial effects are negligibly small. Such an extension is being developed, and in the meantime the early results (Hinch & Acrivos, unpublished) can be used to develop a qualitative theory whose predictions can be compared with experimental observations.

We note first of all that the behaviour of a drop in the hyperbolic flow field whose components with respect to Cartesian axes (x, y, z) are

$$u_z = z, \quad u_y = -y, \quad u_x = 0$$

should not differ greatly from that in the extensional flow considered so far. In fact, it can be shown that, if $\lambda = O(\epsilon^2)$ with $\epsilon \equiv \gamma/G\mu l$, an analysis similar to that of § 3 will apply except for the added, and very serious(!), complication that the shape of the drop will depend now on the azimuthal angle θ, in addition to z. Nevertheless, the dimensional arguments leading to (3.6)–(3.8) are still valid. Hence, if we assume that the deformation curve is qualitatively similar to that in figure 2, we conclude that the condition for drop breakup is, as in § 3,

$$(G\mu a/\gamma)\,\lambda^{\frac{1}{6}} = \text{constant}.$$

This prediction is in very good agreement with Grace's (1971) experimental observations, according to which, at the point of breakup,

$$G\mu a/\gamma = 0{\cdot}1\lambda^{-0{\cdot}16} \quad \text{for} \quad \lambda < 1{\cdot}0.$$

From the theoretical point of view, the effects of inertia on drop breakup should also be similar to those in the extensional flow already considered in §§ 4 and 5, but to date there are no experimental results to test our predictions.

When the impressed flow is the simple shear

$$u_z = y, \quad u_y = 0, \quad u_x = 0$$

and $\lambda \ll 1$, the drop also deforms into a slender shape, more or less aligned with the undisturbed flow. By an order-of-magnitude analysis it can be shown that the small angle of inclination is of the same order as ϵ, i.e. proportional to the slenderness ratio, and hence the undisturbed velocity along the surface of the drop is everywhere $O(\epsilon)$, rather than $O(1)$ as in the cases dealt with so far. Consequently, the normal stress at $r = \epsilon R$ is also $O(\epsilon)$ and therefore it follows from the normal-stress balance that $\epsilon = (\gamma/G\mu l)^{\frac{1}{2}}$. Also, as before, $\lambda = O(\epsilon^2)$. By repeating the dimensional arguments used above in the case of hyperbolic and of extensional flow, we then arrive at the breakup criterion

$$(G\mu a/\gamma)\,\lambda^{\frac{2}{3}} = \text{constant},$$

which is in fair agreement with the corresponding expression

$$(G\mu a/\gamma) = 0{\cdot}17\lambda^{-0{\cdot}55}$$

obtained by fitting Grace's (1971) data for $\lambda < 0{\cdot}1$.

We remark in passing that so far we have considered the behaviour of only a single drop in the applied shear field. It is easy to see, however, that the analysis developed throughout the paper would apply equally well to a moderately concentrated suspension (of the order of 10 % or less in volume concentration) provided that all the drops were slender and were separated from one another by a distance at least $O(\epsilon)$ relative to their length. For, in that case, the inner region surrounding any one drop would be affected by neighbouring drops only through the fact that the presence of other drops alters the effective rate of extension which that one drop feels and hence the shape of its interface would be, to leading order in ϵ and for the given effective rate of extension, the same as that calculated here.

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Appendix A. The shape of the liquid drop near $z = 0$ when ν is not an even integer

The shape of the drop $R(z)$ in the inner region is determined by the normal-stress balance (2.9), which can be solved by an expansion in the small parameter ϵ. Thus we write

$$R = R^{(0)} + \epsilon^2 \log \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \dots . \tag{A 1}$$

The leading term is simply given by (2.5), i.e.

$$R^{(0)} = (2\nu)^{\frac{1}{2}}(1-|z|^\nu), \tag{A 2}$$

while $R^{(1)}$ and $R^{(2)}$ satisfy [cf. (2.9)]

$$\begin{aligned}(z\,d/dz-\nu)(\log \epsilon R^{(1)} + R^{(2)}) = -\{&\log(\epsilon R)[5RR'^2 + 2R^2R'' + zR'^3 + zRR'R''] \\ &+ 2RR'^2 + zR'^3 - \tfrac{1}{4}R'^2 - \tfrac{1}{2}RR'' + \tfrac{1}{2}F_1 R + d(F_2 R)/dz\}_{R=R^{(0)}},\end{aligned} \tag{A 3a}$$

the right-hand side of which becomes, for $\nu > 0$ and $z \to 0$,

$$\{\log(\epsilon R)(2R^2R'') - \tfrac{1}{2}RR'' + \tfrac{1}{2}F_1 R + RF_2'\}_{R=R^{(0)}}. \tag{A 3b}$$

Integration of (A 3), using the expressions (A 17a, b) for F_1 and F_2 derived later on in this appendix, yields

$$R = \frac{1}{2\nu}(1-|z|^\nu) + \frac{\nu-1}{8\nu^2}\epsilon^2\left\{|z|^{\nu-2}\log\left(\frac{2\nu z}{\epsilon}\right) + \alpha_1|z|^{\nu-2} + O(z^{\nu-4}\log z)\right\} + \epsilon^2 R_a(z), \tag{A 4}$$

where
$$\alpha_1 = \log 2 - \gamma - \psi(\nu-1) - \tfrac{1}{2}\pi\cot(\tfrac{1}{2}\pi\nu) + \nu, \tag{A 5a}$$

$$\psi(z) = d[\log\Gamma(z)]/dz, \quad \gamma = -\psi(1) \tag{A 5b, c}$$

and R_a is an even power series in z. In terms of $\tilde{z} = 2\nu z/\epsilon$, (A 4) less the analytic part R_a then becomes

$$R \sim (2\nu)^{-1}\{1-(\epsilon/2\nu)^\nu[\tilde{z}^\nu + \nu(1-\nu)\tilde{z}^{-2}(\log\tilde{z} + \alpha_1)] + \dots\}. \tag{A 6}$$

This expression for R must match with the function $\mathscr{R}_1$ in (2.21) as $\tilde{z} \to \infty$, i.e. $\mathscr{R}_1$ as determined from

$$\frac{\pi\nu}{4}\mathscr{R}_1 = \int_0^\infty\int_0^\infty d\omega\,dt\,\frac{\cos(\omega\tilde{z})\cos(\omega t)}{\omega^2-\Lambda^2}\left\{\mathscr{R}_1\frac{\omega K_1'}{K_1} + \frac{t\mathscr{R}_1' - \mathscr{R}_1}{2}\left[\left(\frac{\omega K_0}{K_1}\right)^2 - 1 - \omega^2\right]\right\} \tag{A 7}$$

must tend to

$$\mathscr{R}_1 = |\tilde{z}|^\nu + c_1|\tilde{z}|^{\nu-2}\log|\tilde{z}| + c_2|\tilde{z}|^{\nu-2} + \tilde{\mathscr{R}}, \tag{A 8}$$

where $\tilde{\mathscr{R}}$ is $o(\tilde{z}^{\nu-2})$.

Now, when (A 8) is substituted into (A 7), we encounter on the right-hand side integrals of the form

$$\int_0^\infty \cos(w\tilde{z})\,F(\omega)\,d\omega\int_0^\infty\cos(\omega t)\,|t|^{\nu-2i}(\log|t|)^j\,dt. \tag{A 9}$$

When ν is not an even integer, we can evaluate the inner integral from the formulae (Lighthill 1958, table 1)

$$\int_0^\infty |t|^\nu \cos(\omega t)\,dt = -\sin\left(\frac{\pi\nu}{2}\right)\Gamma(\nu+1)\,|\omega|^{-\nu-1}, \tag{A 10a}$$

$$\int_0^\infty |t|^{\nu-2}\log|t|\cos(\omega t)\,dt = \sin\frac{\pi\nu}{2}\Gamma(\nu-1)\,|\omega|^{-\nu+1}\{-\log|\omega|+\psi(\nu-1)+\tfrac{1}{2}\pi\cot(\tfrac{1}{2}\pi\nu). \tag{A 10b}$$

The outer integration is carried out by expanding $F(\omega)$ in an ascending power series in ω. This procedure, according to Lighthill (1958), will give the asymptotic value of the integral for large $\tilde{z}$. Equating this to the asymptotic form of the left-hand side of (A 7), we then obtain the consistency relations for the unknown coefficients $c_1, c_2, \ldots$, which are easily found to be

$$c_1 = \nu(1-\nu), \quad c_2 = \nu(1-\nu)\,\alpha_1, \quad \ldots. \tag{A 11}$$

Again we note that, since c_2 is well behaved for $\nu \to 1$, (A 11) apply for all $\nu > 0$ except $\nu = 2, 4, 6, \ldots$.

Using these values for c_1 and c_2 and bearing in mind that

$$2\nu R(z) - 1 = -\epsilon^\nu\{R_1/(2\nu)^\nu + P^{(1)}/2\nu\},$$

we can see that the limiting form of the bubble shape in the inner region, as given by (A 6), is identical to order ϵ^2 with the limiting form of the solution in the singular region, as given by (A 8), provided of course that ν is not an even integer.

We shall next derive (2.23), which leads to the integral constraint on ν. Using the same notation as in (2.23), we have

$$\begin{aligned}
\frac{\pi\nu}{4}\mathscr{R}_1 &= \iint d\omega\,dt\,\frac{\cos(\omega\tilde{z})\cos(\omega t)}{\omega^2-\Lambda^2}\left\{\mathscr{R}_1\mathscr{H} + \frac{t\mathscr{R}_1'-\mathscr{R}_1}{2}\mathscr{K}\right\}\\
&= \int d\omega\cos(\omega\tilde{z})\frac{\mathscr{H}}{\omega^2-\Lambda^2}\int_0^\infty dt\cos(\omega t)\{|t|^\nu + c_1|t|^{\nu-2}\log|t|\\
&\quad + c_2|t|^{\nu-2} + \tilde{\mathscr{R}}\} + \text{corresponding term for } \mathscr{K}\\
&= \int_0^\infty d\omega\cos(\omega\tilde{z})\left\{\frac{1}{\Lambda^2}\left[1-\omega^2\log\omega-\omega^2\left(\gamma-\log 2-\frac{1}{\Lambda^2}\right)\right]\right.\\
&\quad\left.\times\left[-\sin\left(\frac{\pi\nu}{2}\right)\Gamma(\nu+1)\,\omega^{-\nu-1}\right]\right\}\\
&\quad + \int_0^\infty d\omega\cos(\omega\tilde{z})\left\{\frac{\mathscr{H}}{\omega^2-\Lambda^2} - \frac{1}{\Lambda^2}\left[1-\omega^2\log\omega\right.\right.\\
&\quad\left.\left.-\omega^2\left(\gamma-\log 2-\frac{1}{\Lambda^2}\right)\right]\right\}\left\{-\sin\left(\frac{\pi\nu}{2}\right)P(\nu+1)\,\omega^{-\nu-1}\right\}\\
&\quad + \text{terms containing } \omega^{-\nu+1}, \text{ etc.} + \text{terms for } \tilde{\mathscr{R}} + \text{corresponding terms}\\
&\quad \text{for } \mathscr{K}.
\end{aligned} \tag{A 12}$$

The first set of terms cancel with the corresponding terms on the left-hand side of the equation, while the remaining terms become, after some rearrangement,

$$\frac{\pi\nu}{4}\tilde{\mathscr{R}} = \iint d\omega\, dt\, \frac{\cos(\omega\tilde{z})\cos(\omega t)}{\omega^2-\Lambda^2}\left\{\tilde{\mathscr{R}}\mathscr{H} + \frac{t\tilde{\mathscr{R}}'-\tilde{\mathscr{R}}}{2}\mathscr{K}\right\}$$
$$-\sin\left(\frac{\pi\nu}{2}\right)\Gamma(\nu+1)\int_0^\infty d\omega\,\frac{\cos(\omega\tilde{z})}{\omega^{\nu+1}}\left\{\frac{\mathscr{H}}{\omega^2-\Lambda^2}-\frac{1}{\Lambda^2}\left[1-\omega^2\log\omega-\omega^2\left(\gamma-\log 2-\frac{1}{\Lambda^2}\right)\right]\right\}$$
$$+\text{terms containing } \omega^{-\nu+1} \text{ and } \omega^{-\nu-1}$$
$$+\text{terms for } \mathscr{K}. \qquad (\mathrm{A}\,13)$$

Denoting the Fourier cosine transform of $\tilde{\mathscr{R}}$ by $\hat{\mathscr{R}}$, i.e.

$$\hat{\mathscr{R}}(\omega) = \int_0^\infty \tilde{\mathscr{R}}(\tilde{z})\cos(\omega\tilde{z})\,d\tilde{z}, \qquad (\mathrm{A}\,14a)$$

$$\tilde{\mathscr{R}}(\tilde{z}) = \frac{2}{\pi}\int_0^\infty \hat{\mathscr{R}}(\omega)\cos(\omega\tilde{z})\,d\omega, \qquad (\mathrm{A}\,14b)$$

we can rewrite (A 13) in terms of $\hat{\mathscr{R}}$ to obtain

$$\frac{\nu}{2}\hat{\mathscr{R}} = \frac{1}{\omega^2-\Lambda^2}[\hat{\mathscr{R}}\mathscr{H}-\tfrac{1}{2}\mathscr{K}(\omega\hat{\mathscr{R}}'+\hat{\mathscr{R}})]-\Gamma(\nu+1)\sin(\tfrac{1}{2}\nu\pi)$$
$$\times\left\{\left[\frac{\mathscr{H}}{\omega^2-\Lambda^2}\frac{1}{\omega^{\nu+1}}-\frac{1}{\Lambda^2\omega^{\nu+1}}\left(1-\omega^2\log\omega-\omega^2\left(\gamma-\log 2-\frac{1}{\Lambda^2}\right)\right)\right]\right.$$
$$+\frac{\nu-1}{2}\left[\frac{\mathscr{K}}{\omega^2-\Lambda^2}\frac{1}{\omega^{\nu+1}}-\frac{1}{\Lambda^2\omega^{\nu+1}}\left(1+\omega^2\left(1+\frac{1}{\Lambda^2}\right)\right)\right]$$
$$\left.+\text{terms in } \omega^{-\nu+1}\right\},$$

which on rearrangement gives (2.23).

Finally, to complete this appendix, we shall derive the asymptotic forms of F_1 and F_2 as $z\rightarrow 0$, which are required to obtain (A 4). We have from (2.5) and (2.11) that

$$\phi(z) = zRR'' = -\left(\frac{\nu-1}{4\nu}\right)(|z|^{\nu-1}-|z|^{2\nu-1})\operatorname{sgn} z, \qquad (\mathrm{A}\,15)$$

$$g(z) = \frac{1}{4}\frac{d}{dz}(zR^2) = \frac{1}{16\nu^2}(1-2(\nu+1)|z|^\nu+(1+2\nu)|z|^{2\nu}), \qquad (\mathrm{A}\,16)$$

and hence from (2.11) that

$$F_1(z) = -\phi'(z)\log[4(1-z^2)]-(\nu-1)(4\nu)^{-1}[(\nu-1)I_1(z;\nu-2)$$
$$-(2\nu-1)(I_1(z;2\nu-2)], \qquad (\mathrm{A}17a)$$

$$F_2(z) = [\phi(z)-g'(z)]\log[4(1-z^2)]-\phi(z)$$
$$+(8\nu)^{-1}[(3-\nu)I_2(z;\nu-1)-3I_2(z;2\nu-1)], \qquad (\mathrm{A}\,17b)$$

where

$$I_1(z,s) = \int_{-1}^{1}\frac{|t|^s-z^s}{|t-z|}dt$$
$$= z^s\log[z^2/(1-z^2)]-2z^s[\gamma+\psi(-s)+\psi(1+s)+\pi/\sin\pi s]$$
$$+s^{-1}[F(1,-s;1-s;z)+F(1,-s;1-s;-z)], \qquad (\mathrm{A}\,18a)$$

$$I_2(z,t) = \int_{-1}^{1} \frac{|t|^s \operatorname{sgn} t - z^s}{|t-z|}\,dt$$

$$= z^s \log\left[z^2/(1-z^2)\right] - 2z^s[\gamma + \psi(-s) + \psi(1+s) - \pi/\sin \pi s]$$

$$+ s^{-1}[F(1, -s;\ 1-s;\ z) - F(1, -s;\ 1-s;\ -z)]. \qquad \text{(A 18}b\text{)}$$

The above are obtained by expressing I_1 (and similarly I_2) as the sum of three integrals, i.e.

$$\int_{-1}^{1} \frac{|t|^s - z^s}{|t-z|}\,dt = \int_0^1 \frac{t^s - z^s}{t+z}\,dt - \int_0^z \frac{z^s - t^s}{z-t}\,dt + \int_z^1 \frac{t^s - z^s}{t-z}\,dt. \qquad \text{(A 19)}$$

The first two integrals can be readily evaluated while the third becomes

$$z^s \lim_{\delta \to 0} \int_1^{1/z} (t^s - 1)(t-1)^{\delta - 1}\,dt = z^s \lim_{\delta \to 0} \frac{1}{\delta}\left(\frac{1-z}{z}\right)^{\delta} \left\{F\left(-s, \delta;\ 1+\delta;\ \frac{z-1}{z}\right) - 1\right\}$$

$$= s^{-1}F(1, -s;\ 1-s;\ z) + z^s[\psi(1) - \psi(-s)] + z^s \log\left[z/(1-z)\right], \qquad \text{(A 20)}$$

where use has been made of formulae (3.194.1) and (9.132) in Gradshteyn & Ryzhik (1965).

The limiting forms of F_1 and F_2 as $z \to 0^+$ can then be readily derived from (A 15)–(A 17).

Appendix B. The time-dependent equations in the singular region

Since the equations will later be linearized about the steady-state solution, we shall denote the shape of the interface by

$$R^*(\tilde{z}, t) = R_0 + \mathscr{R}^*(\tilde{z}, t), \qquad \text{(B 1}a\text{)}$$

where

$$R_0 = R(0), \quad \mathscr{R}^* = \mathscr{R}(\tilde{z}) + \mathscr{R}'(\tilde{z}, t). \qquad \text{(B 1}b, c\text{)}$$

Here $\mathscr{R}$ is the steady-state solution developed in § 2.3 and $\mathscr{R}'$ is the time-dependent deviation, assumed small compared with $\mathscr{R}$. (Primes below denote small departures from the steady-state values of the corresponding quantities.)

The flow outside $\tilde{r} = R^*(\tilde{z}, t)$ is governed by the creeping-flow equations, subject to the following boundary conditions:

(*a*) that the solution in the singular region matches with the outer and the inner solutions in the appropriate overlap domain;

(*b*) the kinematic condition $\partial R^*/\partial t = u_n$ at $\tilde{r} = R^*$;

(*c*) vanishing shear stress $\sigma_{ns} = 0$ at $\tilde{r} = R^*$;

(*d*) that the normal-stress difference must be equal to the capillary pressure.

To leading order, the above conditions give

$$u^{(0)} = \tilde{z}, \quad v^{(0)} = -\tfrac{1}{2}\tilde{r} + R_0^2/2\tilde{r}, \qquad \text{(B 2), (B 3)}$$

$$\sigma_{ns}^{(0)} = \sigma_{rz}^{(0)} = 0, \quad P^{(0)} = 2 + 1/R_0. \qquad \text{(B 4), (B 5)}$$

These, of course, are just the solutions corresponding to an infinite cylinder of radius R_0.

The next-order solution would provide us with the time-dependent information. We start with (*b*), which is

$$\frac{\partial R^*}{\partial t} = v - \frac{dR}{\partial z}u \quad \text{at} \quad \tilde{r} = R^*,$$

or

$$\partial \mathscr{R}^*/\partial t = -\mathscr{R}^* + v^{(1)} - \tilde{z}\,\partial \mathscr{R}^*/\partial \tilde{z},$$

or

$$v^{(1)} = \partial(\tilde{z}\mathscr{R}^*)/\partial \tilde{z} + \partial \mathscr{R}^*/\partial t. \tag{B 6a}$$

The difference between (B 6*a*) and (2.16*a*) gives the equation for the disturbance velocity (primed):

$$v^{(1)\prime} = \partial(\tilde{z}\mathscr{R}')/\partial \tilde{z} + \partial \mathscr{R}'/\partial t. \tag{B 7a}$$

Similarly, we can show that the shear-stress and normal-stress balances are of the form

$$\sigma_{rz}^{(1)} = 4\,\partial \mathscr{R}^*/\partial \tilde{z}, \tag{B 6b}$$

$$\frac{d^2\mathscr{R}^*}{d\tilde{z}^2} + \left(1 + \frac{1}{\nu}\right)\mathscr{R}^* + \frac{1}{2\nu}\left(2\frac{\partial v^{(1)}}{\partial \tilde{r}} - p^{(1)} + P^{(1)}\right) = 0. \tag{B 6c}$$

The time-dependent disturbance quantities therefore must satisfy (B 7*a*) and the following two boundary conditions (at $\tilde{r} = 1$):

$$\sigma_{rz}^{(1)\prime} = 4\,d\mathscr{R}'/\partial \tilde{z}, \tag{B 7b}$$

$$\frac{d^2\mathscr{R}'}{d\tilde{z}^2} + \left(1 + \frac{1}{\nu}\right)\mathscr{R}' + \frac{1}{2\nu}\left(2\frac{\partial v^{(1)\prime}}{\partial \tilde{r}} - p^{(1)\prime} + P^{(1)\prime}\right) = 0. \tag{B 7c}$$

Notice that (B 7) and (2.16) are the same except for a time-dependent term in the kinematic condition. These equations can be solved by Fourier transforms, as before, to give

$$\mathscr{R}' = \mathscr{C}\cos(\Lambda\tilde{z}) - \frac{P^{(1)\prime}}{2(\nu+1)} + \frac{4}{\pi\nu}\int_0^\infty\int_0^\infty d\omega\, d\eta\, \frac{\cos(\omega\tilde{z})\cos(\omega\eta)}{\omega^2 - \Lambda^2}$$
$$\times\left\{\mathscr{R}'(\eta,t)\frac{\omega\, dK_1(\omega)/d\omega}{K_1(\omega)} + \frac{1}{2}\left(-\mathscr{R}'(\eta,t) + \eta\frac{\partial \mathscr{R}'}{\partial \eta} + \frac{\partial \mathscr{R}'}{\partial t}\right)\left[\left(\frac{\omega K_0}{K_1}\right)^2 - 1 - \omega^2\right]\right\},$$

which is effectively of the same form as (2.18*d*).

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